A Pajarita Puzzle Cube in Papiroflexia

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Abstract

I present the development of a modular origami design based upon the Pajarita, a figure from the traditional Spanish paper-folding art. The work is a modular cube decorated with Pajaritas with the color pattern produced by folding of individual units to be topologically identical but with distinct color patterns on each. The mathematics of the cube-coloring and unit-coloring problems are analyzed. Two unique solutions are found and presented for the easiest possible cube and a solution for the hardest possible cube.

1 Introduction

Origami is the name of the centuries-old Japanese art of paper-folding and has been generally adopted as the name of the paper-folding art, wherever it may be practiced. Japan, however, is not the sole culture with such an art; Spain, in particular, has had a culture of paper-folding, called *papiroflexia*, for well over a hundred years. Its most iconic figure is the *pajarita*, or "little bird" [11]. This shape, described by the philosopher Miguel de Unamuno, has figured heavily in Spanish paper-folding, and is used as the logo of the Asociación Española de Papiroflexia (AEP), the Spanish national society devoted to the art. The Pajarita does not appear in traditional Japanese paper-folding, which suggests a certain independence of creation. It is, however, closely related to the "Windmill" and "Double-boat," both traditional Japanese designs, so there could well be Japanese antecedents to this work that have been lost in the mists of history.

Whether there is a causal connection between Japanese *origami* and Spanish *papiroflexia* is a topic of ongoing research and discussion, but there is definitely a mathematical connection between the two, because the mathematical laws of paper-folding are universal. The

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origami connections to mathematics are diverse, and range from number fields (via folding construction problems [3, 1]) to computational complexity (via the problem of crease assignment [2]) to graph coloring [5]. This last field arises in the design and folding of *modular origami*, in which many sheets of paper are folded into multiples of one or more "units," which are then assembled into various polyhedral structures. The coloring problems arise when one wishes to color the various units in an aesthetically pleasing way, for example, using as few colors as possible while avoiding same-color units meeting at an edge, vertex, or face of the desired polyhedra. The units of modular origami are often relatively easy to fold (individually), but can lead to explorations of many mathematical topics, making this genre particularly suited to mathematics education and/or connecting the general public to mathematical concepts (see, e.g., [6]).

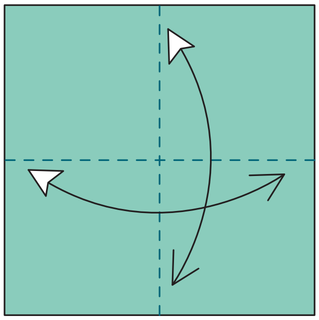
In early 2012 I had the opportunity to attend the AEP's annual convention, and while there conceived of an interesting coloring problem that combines modular origami and the traditional Pajarita in a novel way. Normally, the coloring problems of modular origami assume that one will fold identical units from differently colored sheets of paper. One can, however, fold nearly-identical units from identically colored sheets, but with small differences that give rise to unique colorings for multiple subsets of the units. I recalled a particular modular cube by the late American origami artist Lewis Simon, which, when folded from identical units, displays in its color pattern a distinctive motif strongly reminiscent of both the Japanese windmill and the Spanish Pajarita. This led to an idea: is it possible to fold a 12-unit modular cube in such a way that each face presents the silhouette of a Pajarita, and to do so in a mathematically and origamically interesting way?

The remainder of this paper presents my exploration of this concept. This construction turns out to be possible—in fact, there is an entire family of solutions, each of which I call a "Pajarita Puzzle Cube." The "puzzle" exists on multiple levels: first, there is the puzzle of figuring out how to color the individual units to give a specified Pajarita coloring. Second, there is the puzzle of how to *fold* the individual units so as to achieve the desired coloring. And last, once one has folded the desired set of units, there is the puzzle of physically assembling them into the desired cube. It will turn out that there are multiple solutions to the overall problem that vary widely in difficulty in that final step of assembly, and I will present two of the easiest and one of the hardest varieties of the problem, though I will leave the actual assembly as an exercise for the reader.

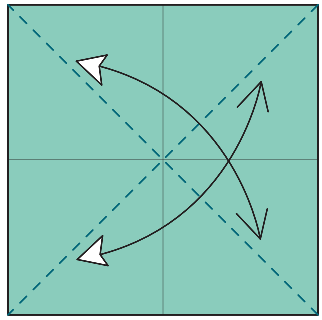
2 The Pajarita

The traditional Spanish Pajarita is the signature figure of Spanish paper-folding. Its origins are lost to history, but it was clearly known in Spain, if not more widely in Europe, by the 19th century [11], and the Spanish poet and philosopher Miguel Unamuno wrote a satire about this figure in 1902. Opinions differ as to how old or widespread it was in the past [9], but it is well known today in Spain and, indeed, in the origami world in general. It is easily folded and instructions are provided in Figure 1.

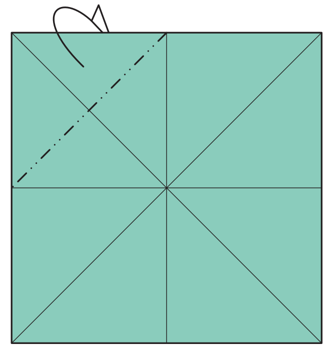
The Pajarita has been adopted as the logo of the Spanish paper-folding society and plays



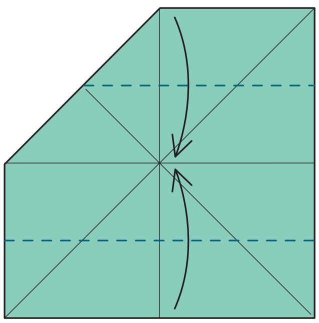
1. Begin with a square. Fold and unfold in half both ways.



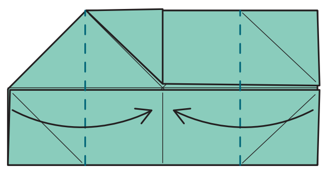
2. Fold and unfold in half both ways along the diagonals.



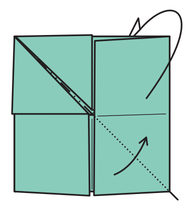
3. Fold the top left corner behind.



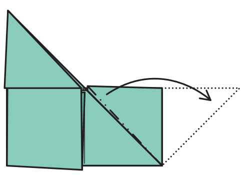
4. Fold the top and bottom edges to the center line.



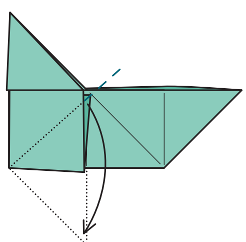
5. Fold the sides in to the center line.



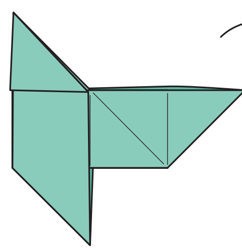
6. Fold the upper right half behind, allowing the bottom right corner to flip upward.



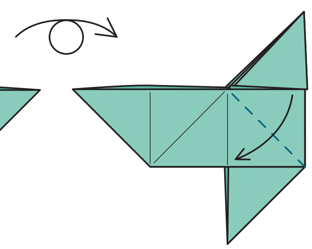
7. Pull a corner all the way out, folding it in half as you do so.



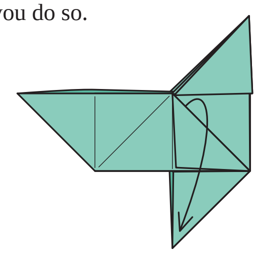
8. Pull another corner out and down in the same way.



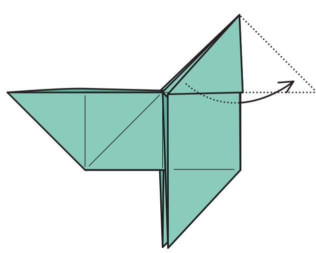
9. Turn the paper over.



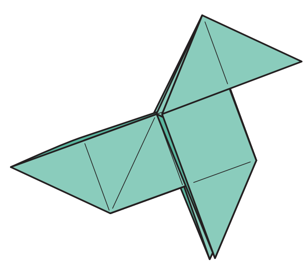
10. Fold the corner down.



11. Grab the raw corner, slide it up, then pull down to match up with the other bottom corner.



12. Pull the corner out from the inside.



13. The finished Pajarita.

Figure 1: Folding instructions for the traditional “Pajarita.”

Figure 2: Lewis Simon's "Modular Cube," folded from 12 squares of two-colored paper.

a symbolic role representing the indigenous paper-folding tradition in Spain that the *tsuru* (crane) does in Japanese origami. For this reason, the Pajarita is a popular motif in Spanish paperfolds by modern creators, and it is the central motif of this exploration.

3 Lewis Simon's Modular Cube

Lewis Simon, an early American creative origami artist, was active in the world of origami for nearly four decades. While he created a wide range of original works, his specialty was modular origami, origami shapes (often polyhedra) in which multiple sheets of paper are folded into "units" that are then locked together, without glue, into a single shape. He, along with Robert Neale [10] and Mitsunobu Sonobe [7], developed many examples of origami modular units based on a concept in which three sequential noncollinear folds form a joint, edge, and joint, respectively, in the desired polyhedron. A more recent example of this genre is Thomas Hull's "PhiZZ" unit, which is perennially popular in the origami/mathematics/education intersection [6].

Regular attendees of meetings of a Los Angeles, California-based folding group, the West Coast Origami Guild, recall Simon showing up every month with a briefcase full of modular designs, often cubes in which the edges and sides were decorated with brilliant and often intricate patterns arising from artful use of the differently colored sides of *kami* (traditional origami paper, which is typically colored on one side and white on the other). Simon died in 1997, but many of his best works are collected by two of his friends, Bennett Arnstein and Rona Gurkewitz, in their book, *Modular Origami Polyhedra* [12]. One of my own favorites, which Simon taught at a WCOG meeting in the early 1980s, is shown in Figure 2.

One of the pleasures of modular origami is that the units themselves are usually relatively easy to fold, making them accessible to people with a wide range of folding experience. The units for this cube are not hard at all to fold, and I encourage the reader to try it out for herself, following the instructions in Figure 3. You will need 12 squares of paper, colored on one side, white on the other, 15 cm or larger.

To make a complete cube, 12 identical units are required. Each unit possesses two tabs and two pockets; units are assembled by inserting the tabs from one unit into the pockets of adjacent units, as illustrated in Figure 4.

Figure 4 shows the units flattened out, but to create a 3-D modular fold, each unit must contain three folds forming 90° dihedral angles, as shown in Figure 3, step 15. The three 90° (partial) creases form a "Z" on the flattened unit. Two units may be assembled while flat, however, to assemble three units at a corner, the Z folds must be in place and the corner made 3-D. A good strategy for assembling the complete cube is to build up one corner from three units, then add additional units successively, always aiming to complete all existing corners before starting a new one.

The completed assembly is deeply satisfying; as the last tab slides into its pocket, there is a relaxation of tension and a dissipation of fragility so that the finished module has a rigidity, stability, and heft that is not present during the assembly process. When it is done, it *feels* done.

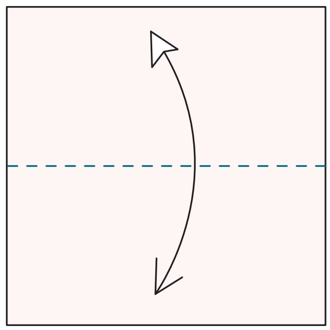
It also has a lovely pattern on its faces, resulting from the colored and white stripes of the units and the over-and-under pattern of the tabs and slots. On each face, there are four identical (but rotated) subpatterns, which are strongly reminiscent of both the traditional Windmill [4] and the Spanish Pajarita.

The pattern arises from the white stripes that are created in steps 5–6 from Figure 3, and many colorful variations are possible by altering these decorative folds (and there are many such examples in [12]). Conversely, by leaving out the "halving" fold of step 5, one can create a mostly white unit that gives rise to a mostly white cube, as illustrated in Figure 5.

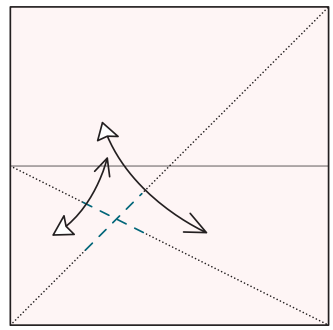
In this case, one can see that the colored triangles in the unit end up giving rise to a colored pinwheel on each face of the cube. One can imagine that by folding units that are mechanically the same but with different color patterns on their surface, one could achieve different patterns on the faces of the resulting cube. If the units are identical and have 180° rotational symmetry, then each face of the resulting cube will be identical and will have 90° rotational symmetry, which is the case here. However, a more interesting question arises: what happens if we don't use identical units? Then we can get different patterns on each face—and it is this question that gives rise to the folding problem of this paper.

4 Customizing a Cube

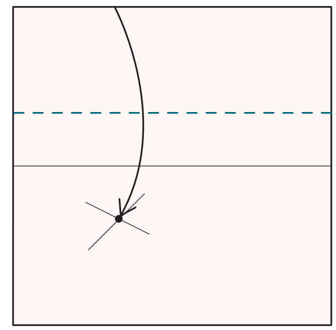
The relationship between the pattern on the cube and the pattern on the individual units is not obvious. First, some portions of the units are covered up by others, so those regions play no role in the finished cube; any pattern on the unexposed regions of a unit serve as "red herrings" when assembling the cube. Furthermore, each unit is exposed, in part, on four *different* faces, two of which are diametrically opposite in the finished cube. So the mapping



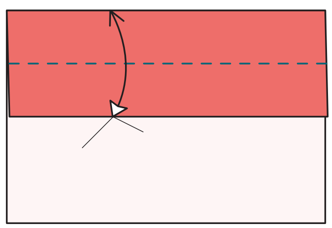
1. Begin with the white side up. Fold and unfold in half.



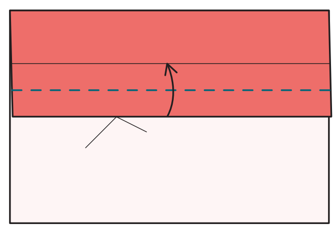
2. Fold and unfold along two diagonals, but only make them sharp where they cross.



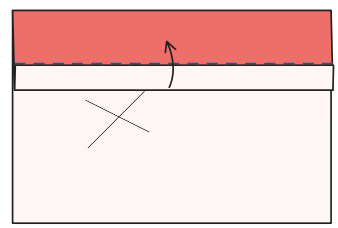
3. Fold the top edge down to the crease intersection.



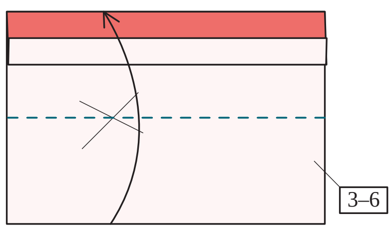
4. Fold the raw edge up to the top edge and unfold.



5. Fold the raw edge up to the existing crease.

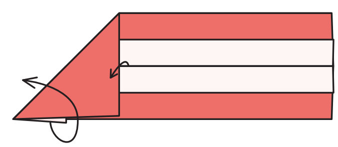


6. Fold the white edge up on the existing crease.

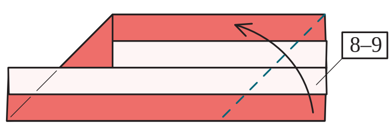


7. Repeat steps 3–6 on the bottom.

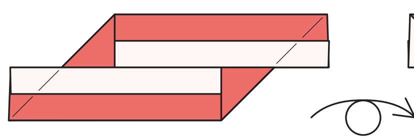
8. Fold the top left corner down, folding through all layers.



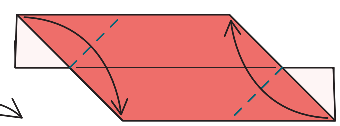
9. Bring the lower layer in front of the folded corner.



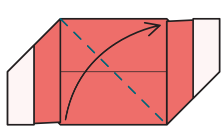
10. Repeat steps 8–9 on the lower right corner.



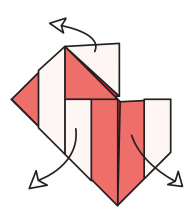
11. Turn over.



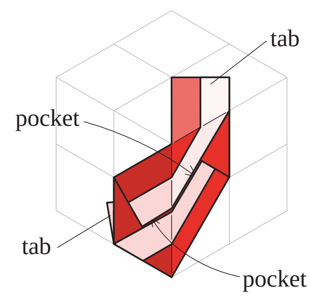
12. Fold the top left corner down and the bottom right corner up.



13. Fold the entire model in half along a diagonal, bringing two corners together.



14. Partially unfold each of the last 3 folds to make a dihedral angle of 90°.



15. This shows tabs and pockets and the orientation of the unit relative to the cube that 12 of them form.

Figure 3: Folding instructions for a single unit of Simon's cube.

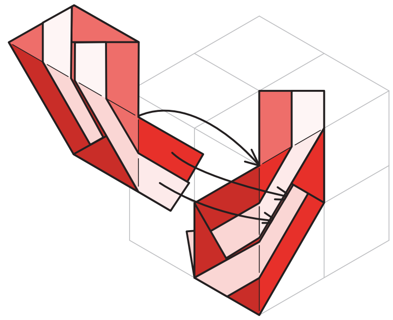


Figure 4: Assembly of two units.

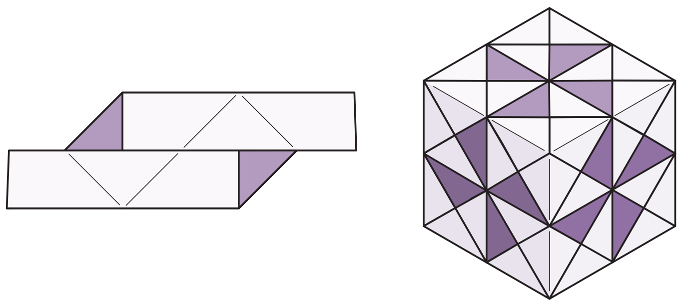


Figure 5: Left: a simplified unit in flat form. Right: the finished cube.

from units to cube faces has a certain amount of mixing going on; this type of mixing is a hallmark of a good puzzle. There's an interesting general problem here: for a given set of patterns on a cube, what are the color patterns that you need on the units to build that cube?

We cannot choose the color pattern arbitrarily, of course; we are constrained by a generic pattern possible with this unit. Each face of the cube consists of four small squares, each subdivided into four triangles, for a total of 16 triangles. If we strive to create no more folded edges beyond those that are created by the mechanical assembly of this type of unit, then in theory, any possible bicoloring of the 96 triangles should be possible with 12 appropriately colored units.

The first step in working out a given coloring would be to determine which regions of the unit are exposed (and where) in a finished cube. One way of making this determination would be to take the generic unit cube of Figure 5, color it entirely on its surface, and then disassemble the units and examine the color pattern on each unit (which would be the same for all units for this thought experiment). Doing so gives the coloring pattern shown in Figure 6.

Each unit exposes eight triangles of the cube (so that the 12 units will expose the total of 96 triangles). In principle, then, one can decide on a color pattern in the finished cube, then project the color pattern back onto the exposed regions of the 12 units. Then, of course, there still remains the not-inconsiderable problem of finding a way of folding each of the 12 desired units with the desired color pattern thereupon. I will presently show the design for

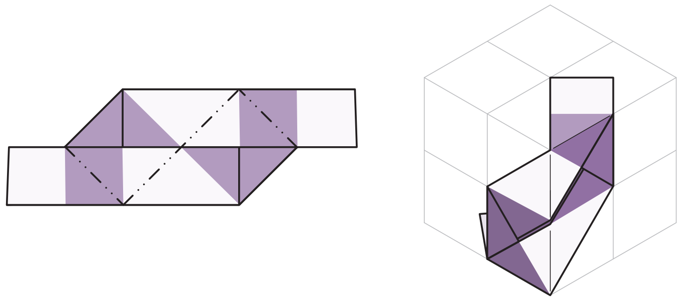


Figure 6: Left: the flattened unit. Colored regions are visible in the finished cube; light regions are hidden. Right: the unit in place in the 3D cube.

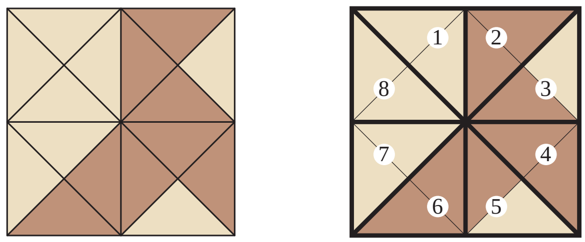


Figure 7: Left: a Pajarita coloring of a 16-triangle face of the modular cube. Right: the coloring divided up into eight darts.

a generic set of units that can be used to achieve any possible coloring, but I am going to do so via a very specific problem, inspired by the original pattern of Simon's module: to create a colored Pajarita upon each face of the folded cube.

5 Pajarita-Colored Cubes

The traditional Pajarita is a 3-D fold with a distinct pattern of overlapping layers, but its silhouette alone is quite recognizable. It maps cleanly to the pattern of 16 triangles that shows up on each face of this modular cube, as shown in Figure 7. I will call any cube whose facial color matches the silhouette of a Pajarita a *Pajarita coloring* of the cube.

The first question that arises in designing a modular Pajarita-colored cube is, how many different ways are there of coloring a cube with Pajaritas? And, beyond knowing how many there are, can we efficiently enumerate and tabulate them?

For each face of the cube, there are 4 possible rotations; thus, there would seem to be $4^6 = 4096$ possible Pajarita-colored cubes. But the Pajarita is handed; it does not have mirror symmetry, so the reflection of a Pajarita will be a different Pajarita. If we define the Pajarita of Figure 7 as a "right-facing Pajarita" (no matter what its rotational orientation is), then its mirror image would be a "left-facing Pajarita," and each face could have 8 different possibilities: 4 left-facing and 4 right-facing, giving a total of $8^6 = 262,144$ possible colorings of an entire cube.

Some of those colorings, however, will be the equivalent to others via an overall rotation of the cube. If we wish to work with distinct colorings, we need to be able to distinguish between colorings that are equivalent under rotation.

One way to do this is to identify each coloring with the position of one or two triangles on the cube face. We can divide the cube face up into 8 triangle pairs, as shown on the right in Figure 7. To distinguish these triangle pairs from the smaller triangles in discussion, I will call these pairs *darts* [14]. (Note that the vertices of a dart are, respectively, a vertex, a midpoint of an edge, and the centroid of a face. This combination is also sometimes called a *flag* [15].) In the figure, dart 6 is the tail of the Pajarita, but in fact, any of the eight darts could be the tail; four of them give left-facing Pajaritas and four give right-facing Pajaritas. Thus, it is sufficient to specify a given Pajarita coloring of a given cube face by specifying which dart corresponds to its tail. (Or to any other part; we could instead use the head, for example, as our signifier of Pajarita position and orientation.)

The same goes for each of the other five faces of the cube; we can specify the Pajarita type and orientation on each face in the same way. A complete Pajarita coloring of the cube can be described concisely by specifying the dart on each face that gives the tail of the Pajarita. There are eight darts per face, so there are 48 darts total on a cube. If we number the darts, 1–48, then a Pajarita coloring can be described as a set of six dart indices, one from each face. Each set of six dart indices specifies a particular arrangement of six Pajaritas on the cube faces.

Now, there are 24 distinct rotations of the cube. Each rotation takes any one dart to some other dart; thus, a cube rotation can be expressed as a permutation of dart indices. A rotation of a Pajarita coloring is a permutation of the set of six darts that define the coloring (transforming those six dart indices into some other set of six indices).

That relationship provides the key to efficiently tabulating the Pajarita colorings that are distinct under rotation. For each coloring, one can find dart index sets corresponding to each of its 24 rotations, sort the indices within each set, then choose the set with the lowest lexicographic ordering. That assigns a unique signature to each Pajarita coloring, using which one can efficiently sort and union the colorings to find a set of all Pajarita colorings that are distinct under rotation. (A link to a *Mathematica*™program for carrying out this analysis is provided at the end of this paper.)

It turns out, then, that there are 11,072 Pajarita colorings of a cube that are distinct under rotations, found by computer tabulation. Figure 8 shows four examples (two views of each cube, to show all faces).

But how do we know that we've truly found them all? As a check on the tabulation, we can use Burnside's Lemma [13] to verify the number of distinct colorings.

Burnside's Lemma states that for a symmetry group G with elements g acting on a colored object, the number N of distinct orbits (which would correspond to distinct Pajarita colorings) is given by

$$N = \frac{1}{|G|} \sum_{g \in G} F(g), \tag{1}$$

where $|G|$ is the order of group G and $F(g)$ is the number of colorings that are invariant under each symmetry operation g. To compute the number of distinct Pajarita colorings, we

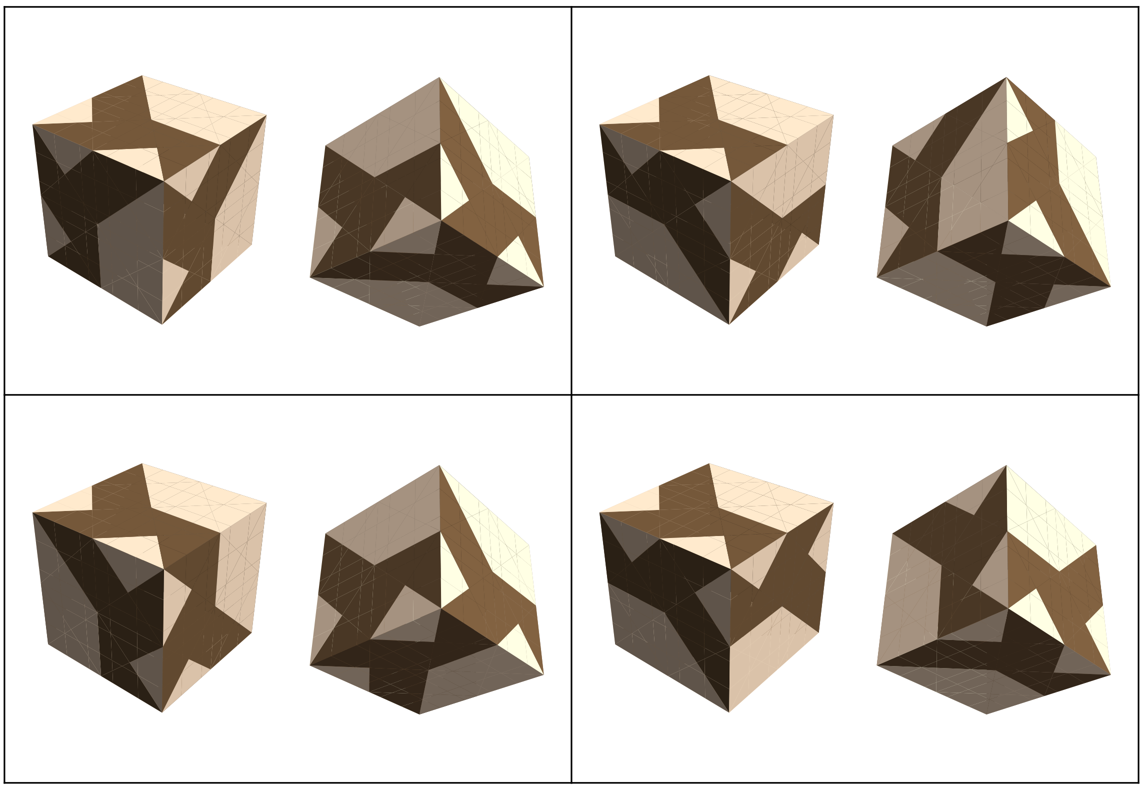


Figure 8: Four of the 11,072 rotationally distinct Pajarita-colored cubes. Each cube is displayed viewing from two diametrically opposite corners so that all faces are visible.

go through each of the rotational symmetry operations of the cube and count the number of "colorings," where by a "coloring" we mean a choice of exactly one of the eight darts on each of the six faces.

For the rotational group of a cube, $|G| = 24$. The rotation operators are:

- I, the identity, for which there are 8^6 possible colorings, since there are eight darts on each of six faces.
- rotations of 180° about any of the six axes through the centers of two opposite edges, for which there are 8^3 possible colorings for each axis.
- rotations of 120° and 240° about axes along any of the four space diagonals, for which there are 8^2 possible colorings for each axis.
- rotations of 90°, 180°, and 270° about any of the three axes through the centers of opposite faces.

For the last set, the facial rotations, we must be a bit careful; our "coloring" is a choice of one of the eight possible darts on each face. Any non-identity rotation of a face about an axis perpendicular to the face must necessarily move the dart on that face to a new position; thus, there are *no* Pajarita colorings invariant under facial rotation (meaning that facial rotations are not symmetry operations of Pajarita-colored cubes). So for these rotations, $F(g) = 0$.

Putting this all together, we find that the number of distinct colorings is

$$N = \frac{1}{24}\left(8^6 + 6 \times 8^3 + 2 \times 4 \times 8^2 + 3 \times 3 \times 0\right) = 11072, \tag{2}$$

which agrees with the count from the computer tabulation.

Now, for each of the 11,072 distinct Pajarita colorings, there is some set of 12 colored units that can be assembled into the given coloring. I would like to find some aesthetic criteria for picking a given coloring. Ideally there would be some criteria that narrows the field down to a single, unique, "best" coloring. As it turns out, there is, in fact, a single "best" coloring—depending on what "best" is taken to be. To define what "best" means, I'll need to start working out the required colorings for the individual units.

6 Unit Colorings

Given a coloring of the cube, it is possible to map the colored triangles of the cube to the visible triangles of the 12 flat units that fold into the given colored cube. There are 96 triangles on the cube and 12 units, so eight colored triangles appear on each flat unit, as was shown in Figure 6.

Each flat unit can be divided into left and right halves, each half containing four exposed triangles, and each triangle can be either white or colored. There are 16 possible ways to color the four triangles of each half, so there are $16 \times 16 = 256$ possible colored units.

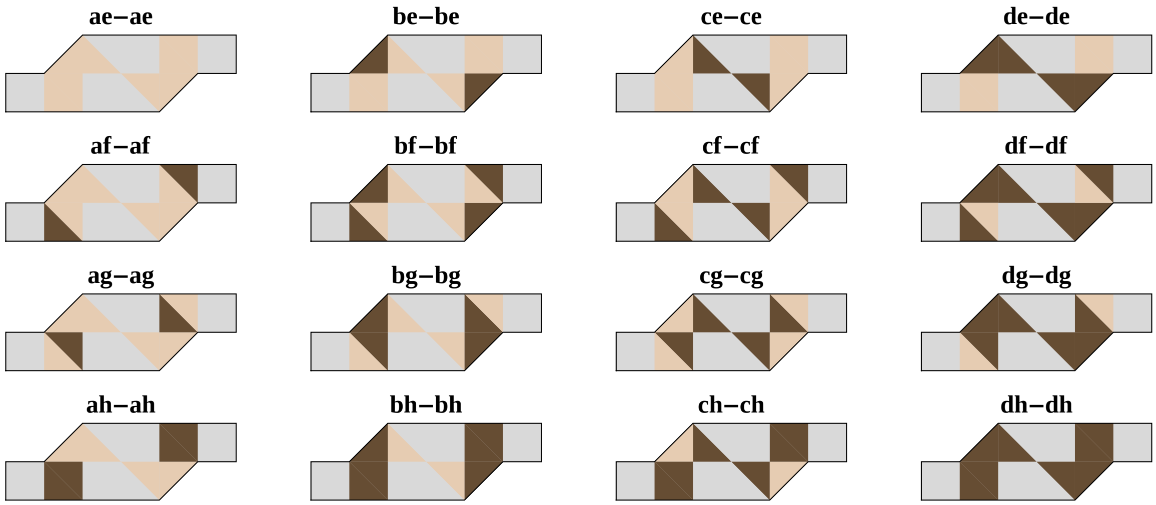


Figure 9: Color schemes for the 16 symmetric flat units. White and dark regions are the regions that are exposed and so must be white or colored; gray regions are not visible in the finished cube and so their color does not matter.

But just as two Pajarita-colored cubes could be equivalent under rotation of the cube, two flat units could be equivalent under 180° rotation of the unit. If we are seeking to find the *distinct* units needed to fold a given cube coloring, then we can count two flat units that are equivalent under 180° rotation to be the same, so the number of distinct possible units is the 16th triangular number, $16 \times 17/2 = 136$ possible unit colorings.

It is convenient to name the possible colorings in a way that makes it easy to associate the corresponding coloring. A flat unit can be divided into left and right halves, and then divided again into top and bottom quadrants, each quadrant containing two exposed triangles and four possible colorings. We can label each of the quadrants by a letter that depends on the four two-colorings of its two triangles: one of **a**, **b**, **c**, or **d** for the upper left and lower right quadrants, and one of **e**, **f**, **g**, or **h** for the lower left and upper right quadrants. A single unit is then described by four letters, two from **a**–**d**, two from **e**–**h**. We can arrange them into a label like **ae-df**, where the four letters describe the four quadrants in counterclockwise order, beginning with the upper left. 16 of the units are rotationally symmetric, and these form a convenient set for displaying all of the possible colorings that can apply to each half of a unit, as illustrated in Figure 9. Note that in the figure, I have colored the *required* (exposed) triangles as white or colored (dark); the intermediate gray regions are those triangles that are hidden in the finished model, and so they can have either color in the folded origami unit. We make use of this freedom when it comes to finding folding sequences for specific units.

Most units, of course, will not be rotationally symmetric like these. But for any given coloring, we can, from the triangles surrounding each edge, work out the specific unit colorings needed for the four quadrants of each unit. Figures 10 and 11 show a colored cube and the units needed to fold it, as well as the positions of each unit in the assembled cube.

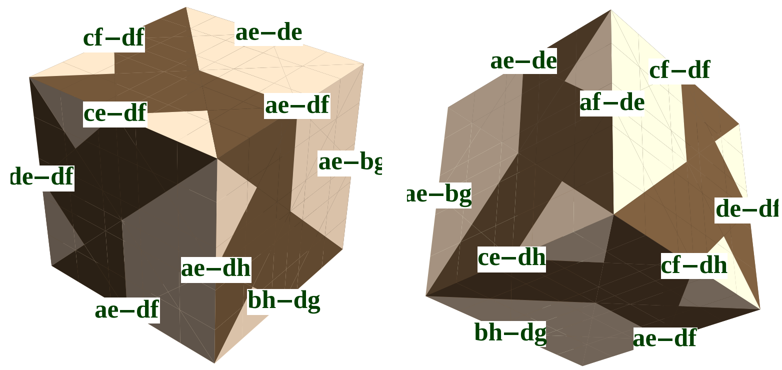


Figure 10: A Pajarita-colored cube and the units needed to fold it.

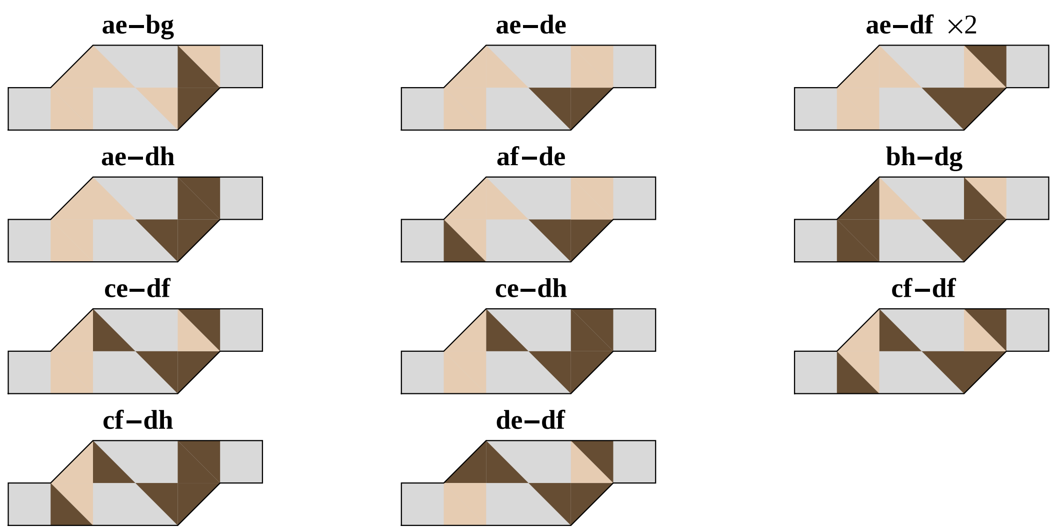


Figure 11: The units needed to fold the Pajarita-colored cube.

Figure 10 shows not only the Pajarita coloring, but also the labels of the units and their positions on each edge. This is a rather useful bit of information to have; if one has only the units but no indication of how they go together, it turns out to be very difficult indeed to figure out how they are assembled to give a desired coloring! On the other hand, that property makes for a very nice puzzle. For such a puzzle, Figure 10, or its equivalent, constitutes the "answer key."

Observe that only 11 units are shown in Figure 11, because one of them (**ae-df**) appears in two places and so has multiplicity 2. The unit multiplicity couples into the difficulty of solving the puzzle. If two units are identical, then they are interchangeable, and that simplifies the puzzle of assembly. If, however, all 12 units are unique, then no two units are interchangeable and the puzzle has maximum difficulty. If a symmetric unit is part of the mix (such as **ae-ae**), that that unit will work in either of its two orientations, which also makes assembly easier. So the easiest possible puzzle would be one with a small number of rotationally symmetric units of high multiplicity, and the most difficult possible puzzle would be one in which there are 12 units, none of which are symmetric, that can be assembled to make a Pajarita-colored cube in exactly one way.

7 Unit Foldings

As noted earlier, there are 136 possible distinct color schemes for flat units, but every unit can be divided into two halves and there are only 16 distinct half-unit color schemes. But the problem is even smaller than that: each half-unit consists of two quadrants, and there are only four possible colorings per quadrant. So we can create any four-quadrant unit if we have a folding method that allows the four quadrants of a unit to be folded independently; then only eight distinct quadrant folding methods would be needed.

In order to create a modular cube from the unit, one must fold a unit that possesses the desired color scheme and also has tabs and pockets in the right places—basically, the same places as the original Simon Cube (or the generic variant).

Another important consideration is efficiency of paper utilization: if the unit is too thick and has too many layers, it will be difficult to fold and assemble the units. If we take the square tabs at the ends of the units to be unit squares, then the Simon and generic units are folded from a 4×4 grid of squares. In order to get enough additional paper to create color patterns, it is necessary to go to a larger grid, but as it turns out, one needs only one additional square of width/height for each quadrant to obtain all the desired colorings. So a 6×6 square gives sufficient paper for folding all of the color schemes.

Every unit starts out the same way, using the folding sequence shown in Figure 12.

The procedure for folding any unit specified by its 4-character label is to fold the top and bottom quadrants on the left side, corresponding to the first two letters; then rotate the unit 180° and fold the top and bottom quadrants on the (new) left side corresponding to the second two letters. The mapping of which letters go with which quadrants is shown in Figure 13.

Folding sequences for upper-left quadrants of type **a**–**d** are shown in Figures 14–17. (Note

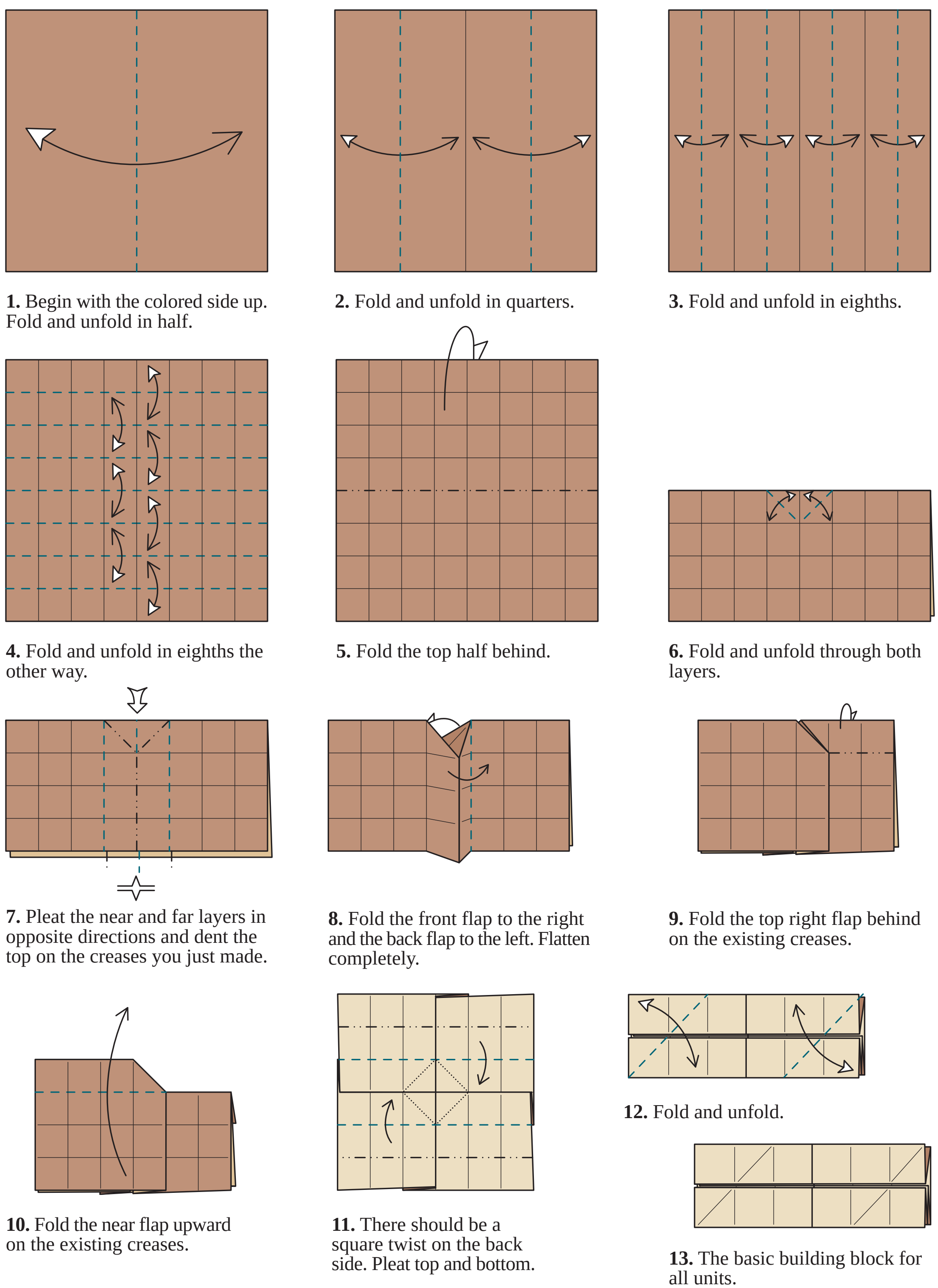


Figure 12: Folding sequence for the building block for all possible units.

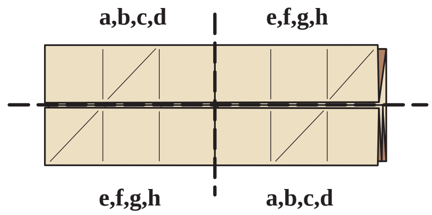


Figure 13: Mapping of quadrant types to regions of the building block.

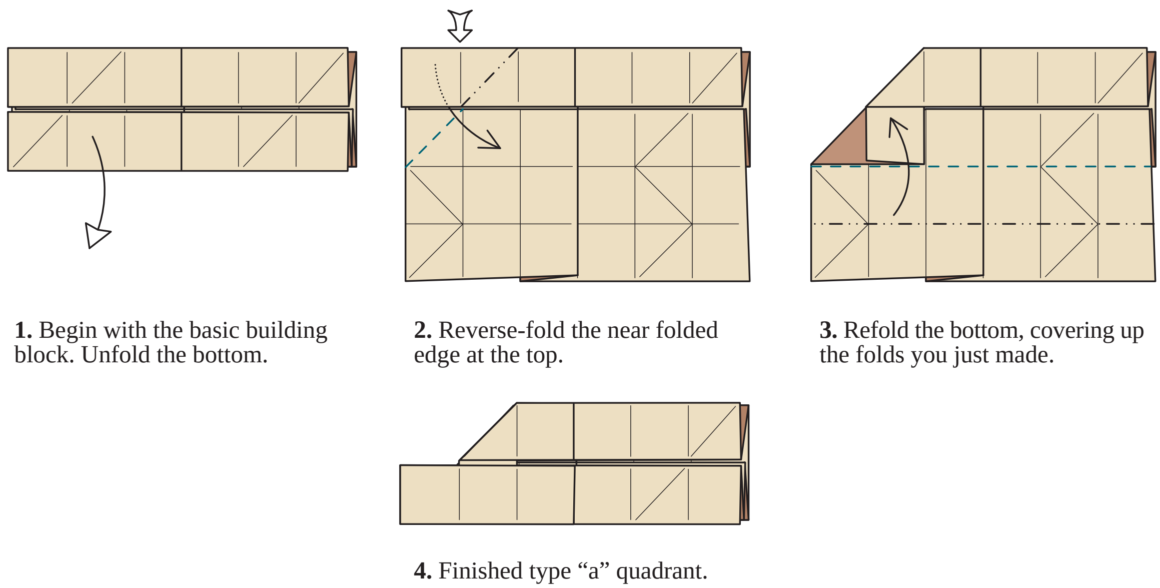


Figure 14: Folding sequence for a type **a** quadrant.

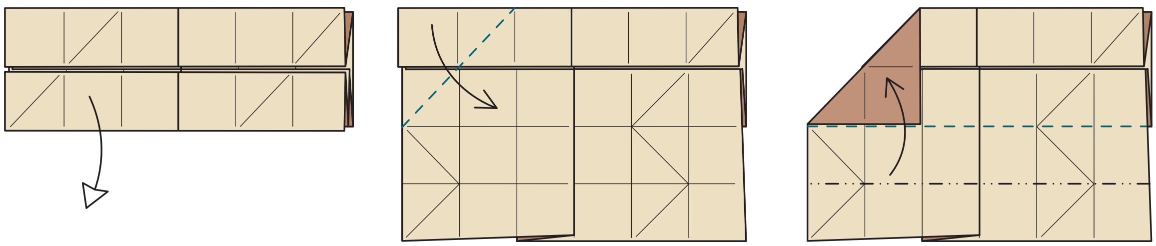
that in order to keep the folding sequences tractably short, I have used several terms common in the world of origami to describe combination folds: "reverse fold," "Elias stretch," and so forth. See, for example, [8] for a detailed explanation of these maneuvers.)

Folding sequences for the lower-left quadrants of types **e**–**h** are shown in Figures 18–21.

Each quadrant is shown folded in isolation here, but of course, you will need to fold four quadrants together in a real unit. Generally, the top and bottom quadrants can be folded nearly independently of one another, but after folding the pair from one side and rotating to fold the other two, some care must be taken when folding the last two quadrants to avoid disturbing the first two folded.

All of the quadrant types give rise to the desired coloring and have the tabs and pockets required to assemble the units. However, quadrant **g** is slightly different from all the rest; its tab is triangular, rather than square, so its joint is not as secure as the others. That is an aesthetic deficiency in this type of unit. That establishes an aesthetic criterion for colorings: all else being equal, it would be desirable to avoid use of quadrant **g**.

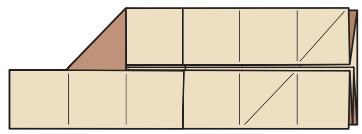
Even if a unit set includes quadrant **g**, there is a way of avoiding it that sometimes works: that is to invert the parity of the paper, i.e., start with the white side up when folding the



1. Begin with the basic building block. Unfold the bottom.

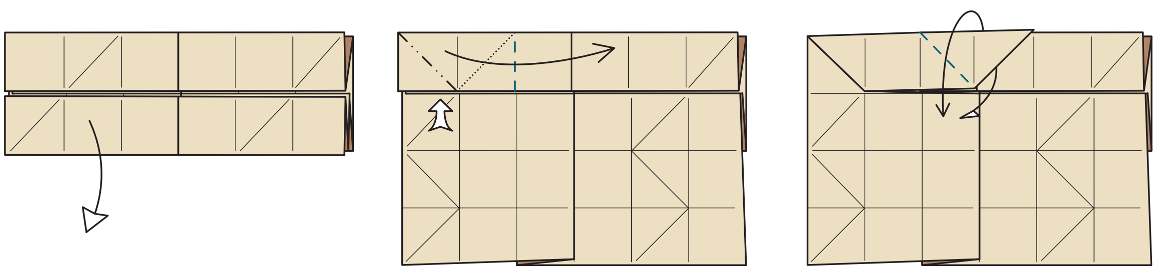
2. Valley-fold the top left corner through all layers.

3. Refold the bottom, covering up the folds you just made.



4. Finished type "b" quadrant.

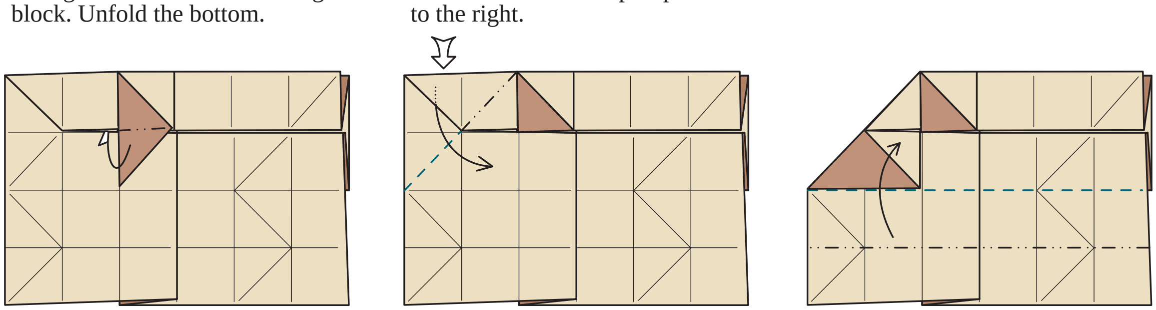
Figure 15: Folding sequence for a type **b** quadrant.



1. Begin with the basic building block. Unfold the bottom.

2. Elias-stretch the top flap over to the right.

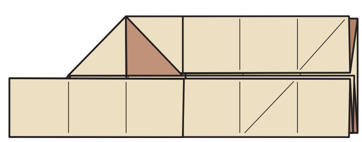
3. Outside reverse-fold the corner.



4. Tuck the corner underneath all of the white edges.

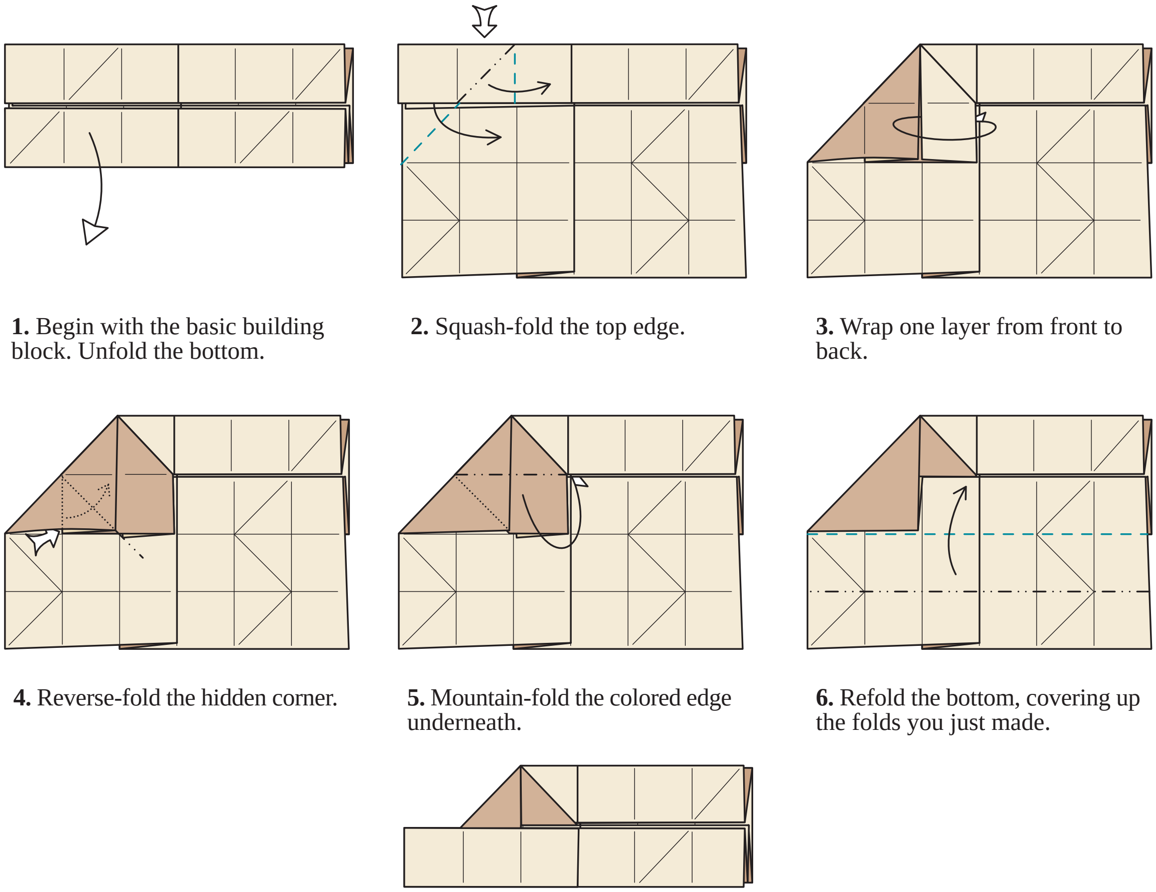
5. Reverse-fold the corner.

6. Refold the bottom, covering up the folds you just made.



7. Finished type "c" quadrant.

Figure 16: Folding sequence for a type **c** quadrant.



1. Begin with the basic building block. Unfold the bottom.

2. Squash-fold the top edge.

3. Wrap one layer from front to back.

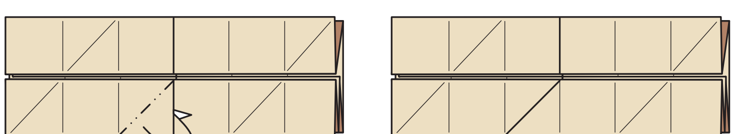
4. Reverse-fold the hidden corner.

5. Mountain-fold the colored edge underneath.

6. Refold the bottom, covering up the folds you just made.

7. Finished type "d" quadrant.

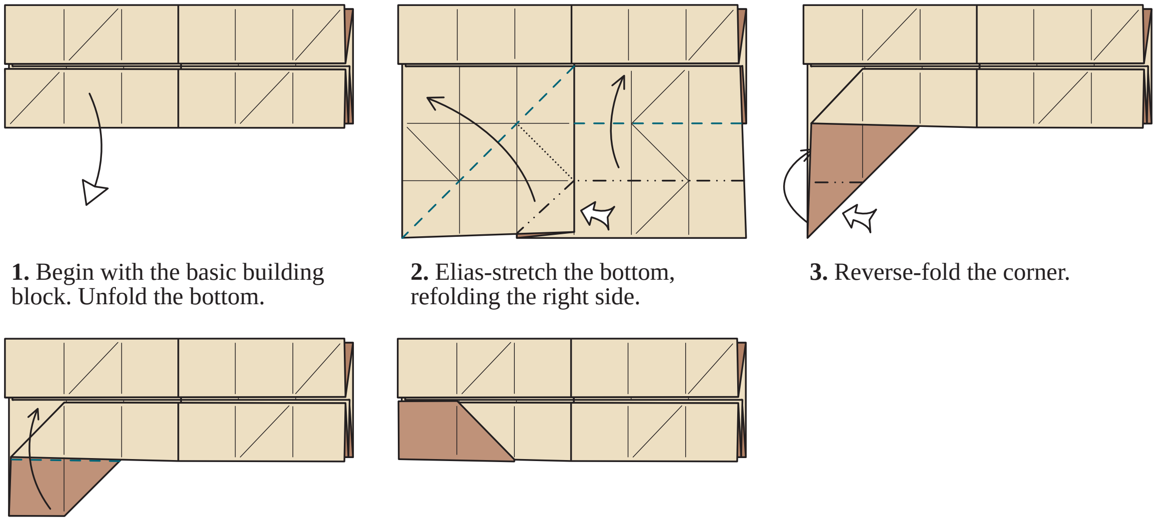
Figure 17: Folding sequence for a type **d** quadrant.



1. Begin with the basic building block. Fold a corner underneath.

2. Finished type "e" quadrant.

Figure 18: Folding sequence for a type **e** quadrant.



1. Begin with the basic building block. Unfold the bottom.

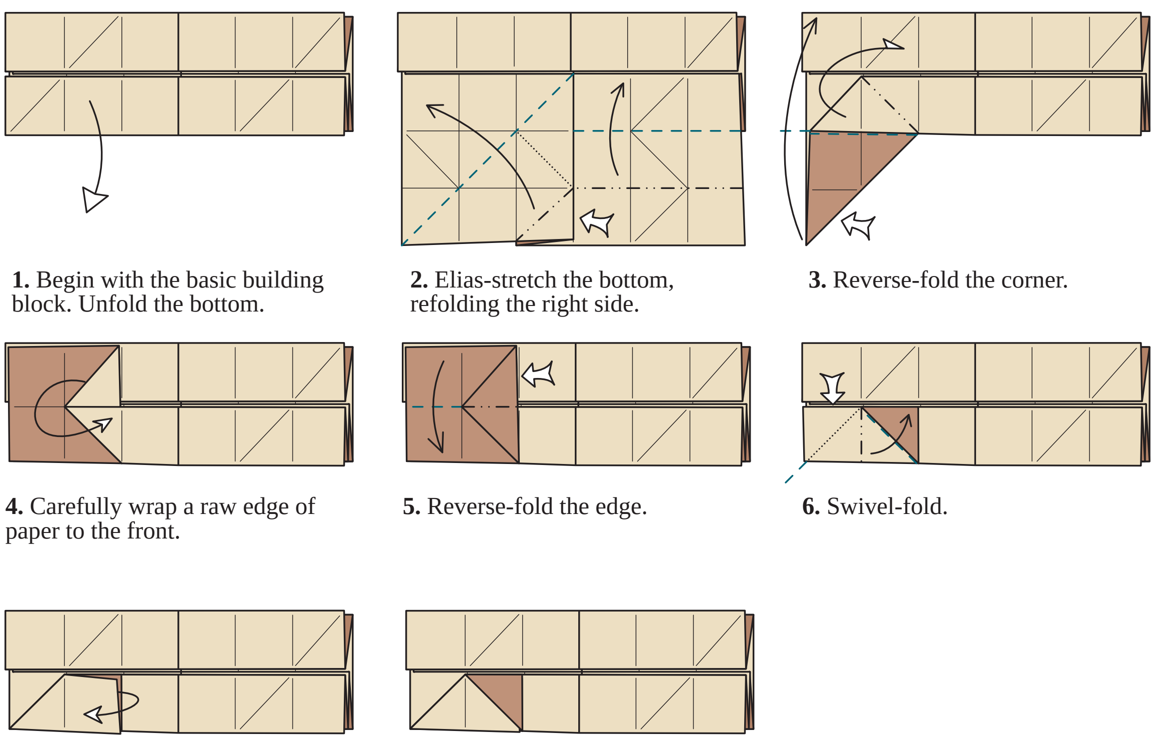
2. Elias-stretch the bottom, refolding the right side.

3. Reverse-fold the corner.

4. Fold the colored tab upward.

5. Finished type “f” quadrant.

Figure 19: Folding sequence for a type **f** quadrant.



1. Begin with the basic building block. Unfold the bottom.

2. Elias-stretch the bottom, refolding the right side.

3. Reverse-fold the corner.

4. Carefully wrap a raw edge of paper to the front.

5. Reverse-fold the edge.

6. Swivel-fold.

7. Bring one layer in front of the corner.

8. Finished type “g” quadrant.

Figure 20: Folding sequence for a type **g** quadrant.

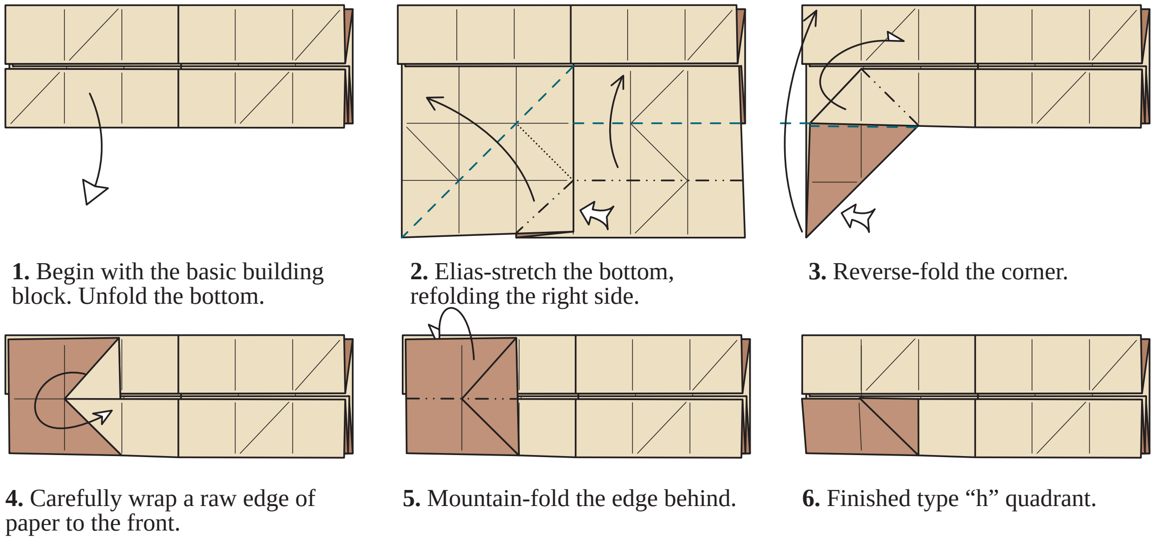


Figure 21: Folding sequence for a type **h** quadrant.

basic building block. This would have the effect of inverting the color parity of the exposed regions (as well as changing colors for some hidden regions), in a way that can be summarized as follows:

$$\begin{array}{ccc} \mathbf{a} & \leftrightarrow & \mathbf{d} \\ \mathbf{b} & \leftrightarrow & \mathbf{c} \\ \mathbf{e} & \leftrightarrow & \mathbf{h} \\ \mathbf{f} & \leftrightarrow & \mathbf{g} \end{array}$$

Inverting the color of the unit will change any **g** quadrants to **f** quadrants. However, it will also perform the reverse as well, so any unit whose coloration is of the form ***f–*g** would be immune to this dodge.

But perhaps we will get lucky. Let's find out.

8 Significant Colorings

I set up a *Mathematica*™notebook to tabulate the possible Pajarita colorings and decompose each coloring into its 12 units. Out of the 11,072 distinct colorings, two require two types of unit; nine required three types of unit; 1822 required 12 different units; the rest were somewhere in between.

Also, somewhat surprisingly, although there are 136 theoretical units, all possible colorings can be folded using only 55 of them, due to constraints within the Pajarita shape. For example, no matter how the pattern in Figure 7 is rotated or flipped, if the two small central triangles of a square quadrant are white, then the two outer small triangles of that quadrant are also white, eliminating the combinations **ag** and **ah**.

If we look at the 11 colorings that require only two or three types of unit, those will, in general, be easier to fold in practice, because identical units can be swapped for one another.

In fact, we can quantify how the unit multiplicity (and other things) leads to ease or difficulty in assembling a given Pajarita-colored cube.

Let us first quantify how hard it could get. If we are handed a set of 12 units that are to be used to fold a given orientation of a cube, there are 12! assignments of units to the edges of the cube. Each unit can also be placed into two different orientations, so there are $12! \times 2^{12} = 1,961,990,553,600$ possible arrangements of the units. In the worst-case scenario, if there is only one solution, then among all of those arrangements, only 24 (corresponding to the 24 rotations of the unique solution) would give rise to a valid Pajarita coloring. The odds of any randomly chosen arrangement being the desired solution are, then, rather small.

However, the worst-case scenario does not always apply. There are various symmetries that can give rise to multiple ways of assembling units successfully:

- A set of units could be assembled into two different, but valid, Pajarita colorings;
- For any unit that has multiplicity $m > 1$, there are $m!$ permutations of those units that will give the same coloring;
- Any unit that has 180° rotational symmetry will work in either of 2 orientations.

So it is possible that a given set of units has multiple solutions that are valid Pajarita colorings. Considered as a puzzle, the more solutions there are for a given set of units, the easier the assembly problem becomes.

As it turns out, among the unit sets corresponding to the 11,072 distinct colorings, the number of possible valid unit arrangements for a given unit set ranges from 1 to 106,168,120. These last sets are in some sense the easiest possible sets to assemble, because the individual units can be assembled into valid Pajarita colorings in the largest possible number of ways (irrespective of color matching). There are two maximally easy sets, which are shown, along with their colorings, in Figure 22 and 23.

Both of these are highly symmetric (which, of course, gives rise to the simplicity of solution). There are only three types of unit: two with multiplicity 3, one with multiplicity 6. It is relatively easy to assemble the set of 12 units into the desired cube, even without the “answer key;” in fact, just knowing the symmetry of the coloring pattern is sufficient to infer the positions of all of the units within the cube, and, since all units are rotationally symmetric, their rotational orientation is irrelevant. A folded example of Figure 22 is shown in Figure 24.

And we get a little bit lucky with the first pattern of Figure 22: it does not contain the undesirable quadrant **g**. So there is a unique solution for “easiest unit set that doesn’t include **g**.”

The second solution, Figure 23, does contain **g**, but it doesn’t contain **f**. So the strategy already mentioned of parity inversion will work. Instead of starting with the colored side up and folding **ae-ae**, **bg-bg**, and **dg-dg**, one could start with the white side up and fold **dh-dh**, **cf-cf**, and **af-af** units, respectively; these, then, would have inverted color patterns, and could be assembled into the desired cube coloring (with the proper color parity).

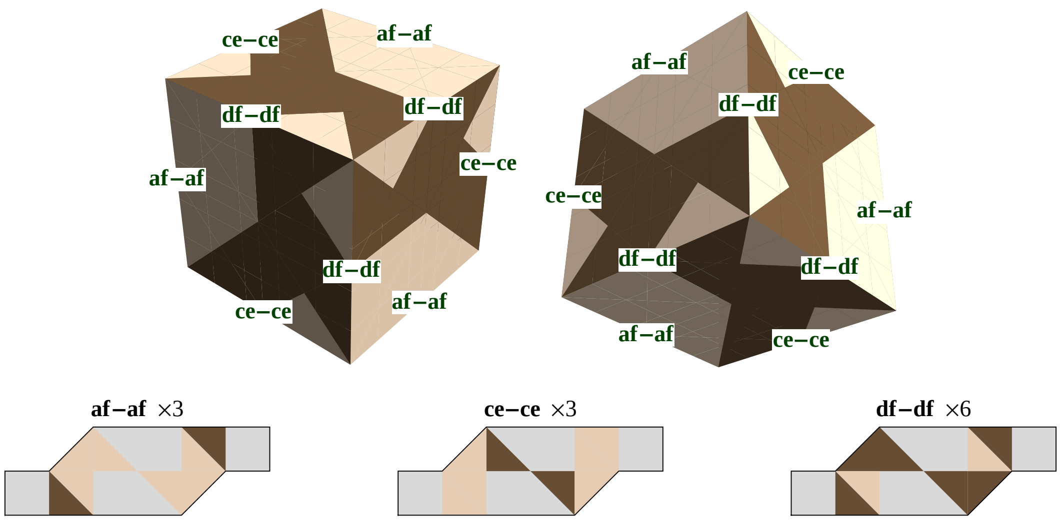


Figure 22: An easiest Pajarita Puzzle Cube. Top: two views of the colored cubes with edges labeled with the units that go on those edges. Bottom: the unit colorings.

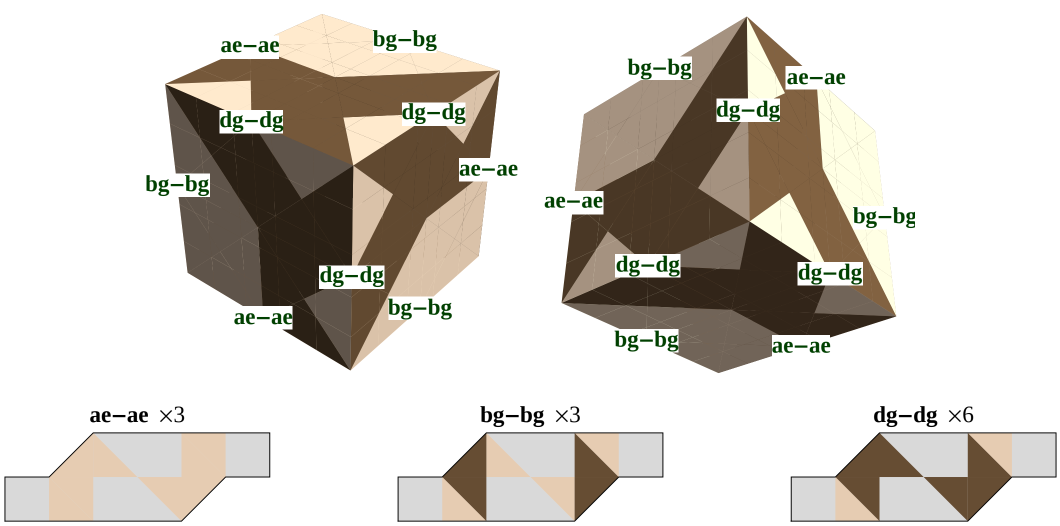


Figure 23: The other easiest Pajarita Puzzle Cube. Top: two views of the colored cubes with edges labeled with the units that go on those edges. Bottom: the unit colorings.

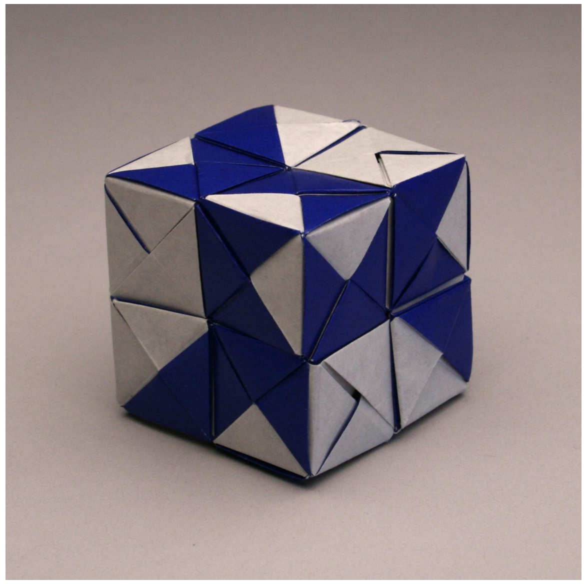


Figure 24: Folded model of the “easy” Pajarita Puzzle Cube.

We note in passing that this solution was chosen as the “easiest” by our measure of ”having the most possible valid arrangements,” independent of the color pattern. In practice, one would of course use the color patterns on the units and faces to assist in getting the individual units in the right places—although, due to the distribution of color patches across units and edges, even this “easiest” unit still poses some challenge.

9 The Hardest Pajarita Puzzle Cube

The three-types-of-unit solutions are challenging to fold, but, relatively speaking, they are still simple because of the high symmetry. Far harder would be an assembly with 12 different types of unit, none with 180° rotational symmetry. There are 321 of those. These patterns would be the hardest to assemble (without an answer key), because there is only one way each set of units can be assembled. Most of them use all eight quadrant types, but it turns out that there is exactly one solution that uses only seven. Better yet, the missing quadrant is the problematic quadrant **g**! So we get lucky again: there is exactly one possible candidate for “hardest possible puzzle that doesn’t use **g**,” and that is shown in Figure 25.

The reader will observe, perhaps, that I have not labeled the edges with their corresponding units. I will leave the actual folding and assembly of this cube as a challenge for the reader.

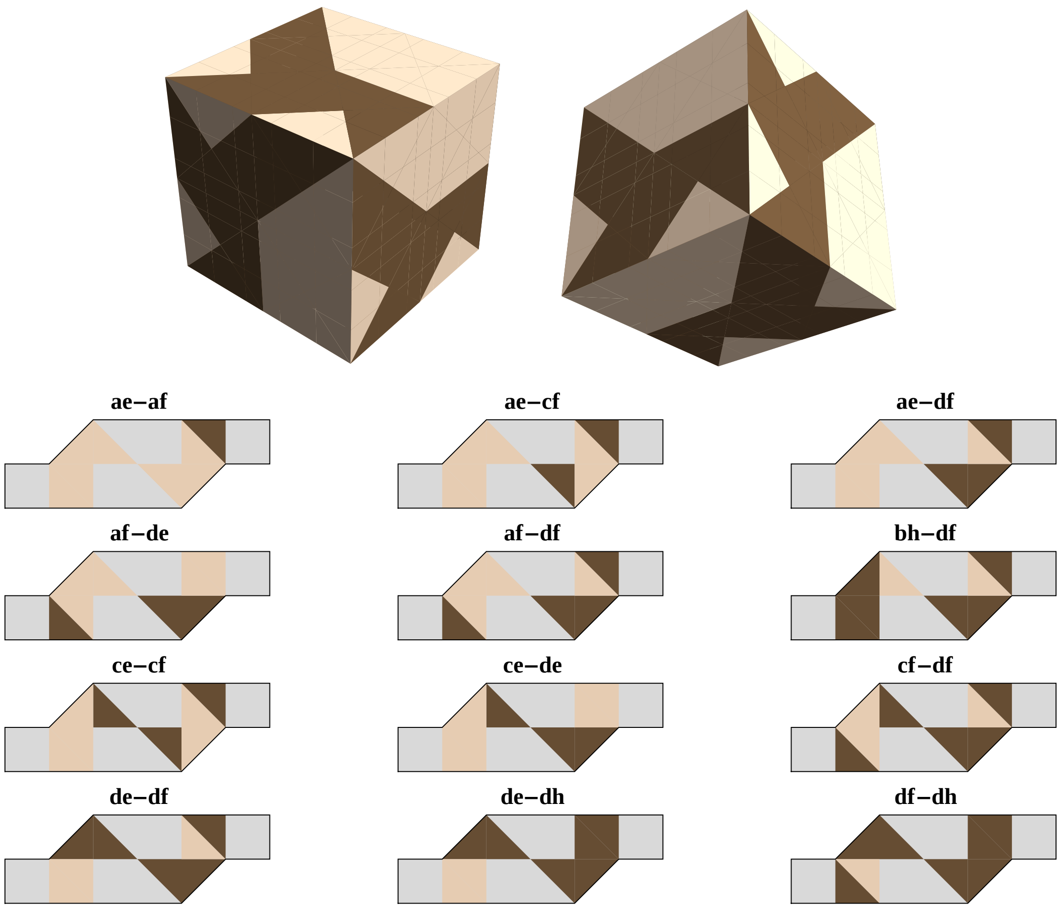


Figure 25: The hardest Pajarita Puzzle Cube. Top: two views of the colored cubes with edges labeled with the units that go on those edges. Bottom: the unit colorings.

10 Conclusions

To conclude, I have described the evolution of an origami (or papiroflexia) figure that, to me, at least, combines equal parts of origami, history, and mathematics. The development of the design is an interesting mathematical (and programming) challenge. The development of folding sequences for the unit quadrants provided an interesting folding challenge. And then all readers can share in the challenge of folding the units and assembling them into the finished Pajarita-colored figure.

There's more that can be done with this concept. My colleague Jason Ku has pointed out that there is an additional degree of freedom possible in the coloring: we could selectively invert the color pattern on any given face, converting that face's Pajarita from colored-on-white to white-on-colored. This change would give rise to an even larger family of color patterns. Since the eight quadrants cover all possible facial colorings, it should still be possible to fold any such color pattern using the quadrant folding methods described in this article.

And of course, one could branch out even further, folding arbitrary color patterns on each face—provided, of course, that they fit into the dissection of each face into the 16 triangles. There are many fruitful opportunities for further development of the concept. Because of the historical connection of the Pajarita and Spanish paper-folding, I particularly like this motif, and it was a pleasant development to find aesthetic solutions for both the easiest and hardest possible colorings.

For readers who would like to explore this analysis further (or wish to peek at the answer key for the hardest puzzle), a *Mathematica*™8 notebook containing my analysis may be downloaded from `http://www.langorigami.com/science/computational/ppcube/pajarita_puzzle_cube_ma8.nb`. There are two solutions described in the notebook that require only two types of edge unit and whose difficulty rating is only slightly higher than the two easy solutions presented here; interested readers might enjoy working out the folding sequences and assembly for the two.

Acknowledgements

I would like to thank the late Lewis Simon for his many demonstrations at West Coast Origami Guild meetings in the 1980s, Sr. Vicente Palacios for sharing his research into the Pajarita and papiroflexia, Bennett Arnstein for permission to reproduce instructions for Simon's modular cube, and Jason Ku for useful suggestions. I would also like to thank the Editor and two anonymous referees for their helpful comments and recommendations.

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