Origami Burrs and Woven Polyhedra, Part II

by Robert J. Lang

In Part I, I introduced the idea of woven origami polyhedra (the concept, of course, had already been discovered by others before me) and described a technique for constructing them based on rotating the edges of uniform-edge convex polyhedra. By applying this technique to the dodecahedron, I found compounds of woven triangles, pentagons, and rediscovered Hull's Five Intersecting Tetrahedra. In this section, I'll apply the technique to other polyhedra.

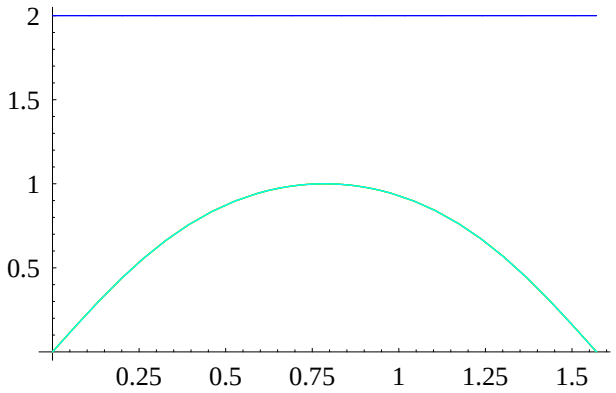
Since I wrote this, I've developed a different approach that allows one to systematically enumerate all woven polyhedra. I'll describe this new approach in a third installment in the future.

Since writing the last installment, I had it pointed out to me that the name "woven polyhedron" was not very accurate, because by rights a "polyhedron" should be a single connected structure. What I was after was a compound of polyhedra, like Hull's FIT. After some thought, I came up with the idea that since "polyhedron" means "many faces," and I was making something that was made up of "many polyhedra," the name "polypolyhedra" would be an appropriate descriptor. So that's what I'll use for the rest of this article.

What I was after was compounds of polyhedra (polypolyhedra) in which the polyhedra are linked together by penetrating each other with their edges intersecting. If we require that the polypolyhedron be built from multiple instances of a single edge unit, then the polypolyhedron must be uniform-edge. There wasn't (to my knowledge) any listing of uniform-edge polypolyhedra, so I decided to try to construct them. The scheme I developed, which I described in my last article, was pretty simple. Start with a uniform-edge polyhedron, lengthen and rotate each edge about its midpoint like a propellor. When the edges hits other edges, examine the result to see if it's an interesting polypolyhedron.

Although I had developed a pretty nice origami modular (the 10×1×3) by applying my rotating-edges algorithm to the dodecahedron, I wasn't done by any means, because I had only plumbed this one polyhedron. I still had 4 other uniform-edge polyhedra to go. Who knows what other fantastic structures might lie within them? I set about to examine them all.

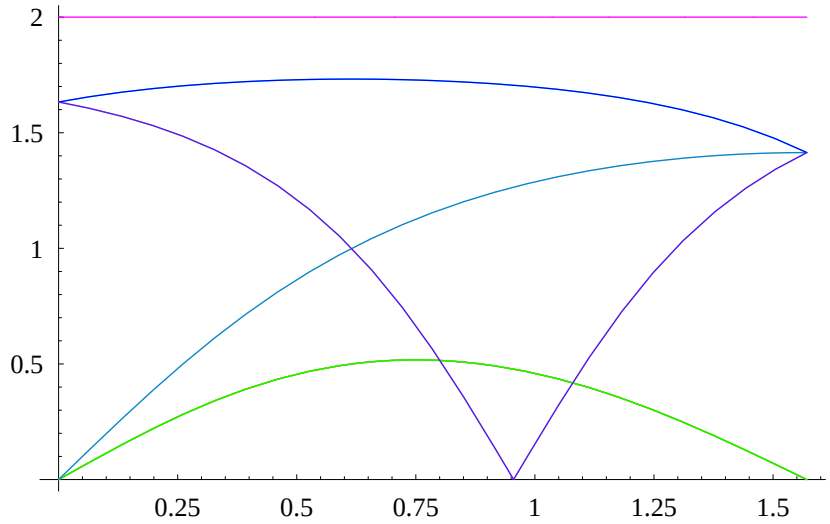
The simplest of the uniform-edges is the tetrahedron. I took the same approach: allow each edge to rotate about an axis connecting its midpoint to the center of the polyhedron. I then plotted the distances from a reference edge to each of the other 5 edges, as shown below.



Between 0 radians (the original tetrahedron) and $\pi/2$ radians (the dual, which is another tetrahedron), there are no zero crossings for any of the edges. So the tetrahedron does not offer any possibilities for polypolyhedra.

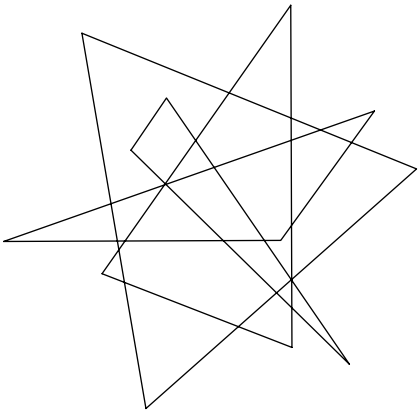
The preceding statement is not precisely correct: there are a couple of polypolyhedra with tetrahedral symmetry, but they have more than six edges. We'll meet them all in Part III.

My next stop was the octahedron. The octahedron has 12 edges. Again, the same process: rotate each edge, and plot its distances from the other 11 edges.

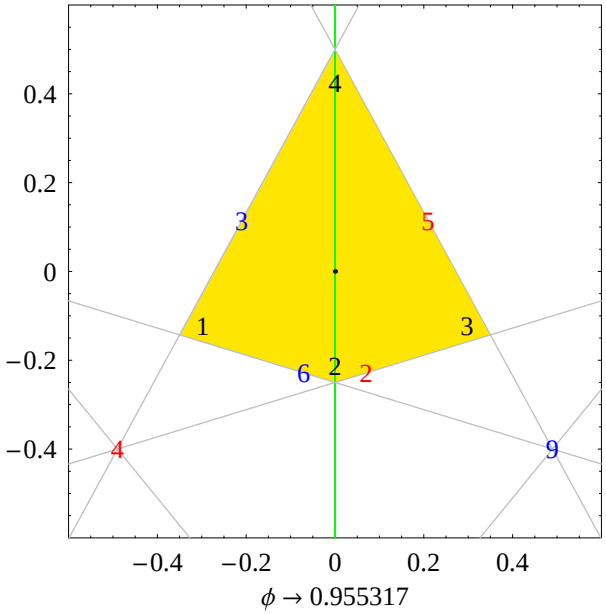


This had a single crossing between 0 radians (the original octahedron) and $\pi/2$ radians (its dual, the cube), located at a rotation angle of ϕ=0.955317. The skeleton corresponding to this crossing is shown here.

$\phi \to 0.955317$



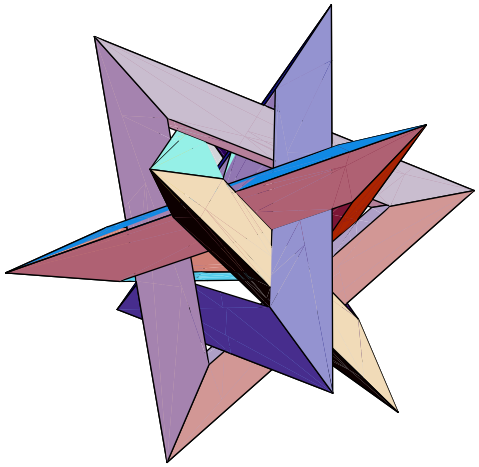
So the octahedron, too, has a polypolyhedron hidden within it: in this case, it is a compound of 4 triangles, or using my notation, a 4×1×3 polyhedron. For this polyhedron, as I did for the dodecahedron, I could plot the cross section to determine the largest possible cross section for each stick.



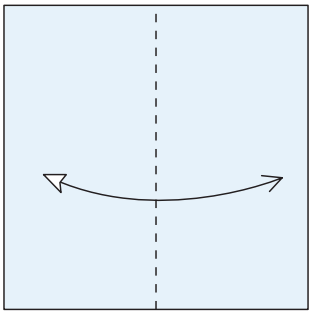
Again, the maximum cross section was a kite shape, and again, two of the sides were nearly hidden and thus could probably be eliminated from the stick. Doing this gave the following shape for the unwrapped stick:

0.214286	0.281718	0.785714	0.281718
0.	0.140859	1.	0.140859
0.214286	0	0.785714	0

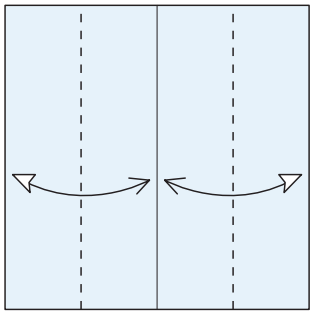
And the completed polypolyhedron is this.



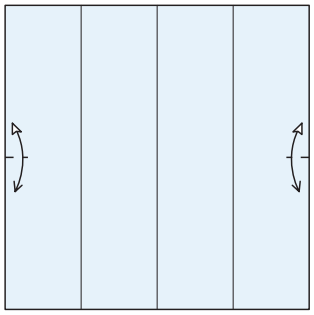
So how to fold the stick? The length-to-width ratio of the unwrapped stick pattern (above) is 3.55:1, while the apex angle at each side is 33.3°. So this unit could actually be folded from Francis Ow's edge unit by merely changing a few of the dimensions and angles involved. Now one could, of course, build the unit by cutting a 3.55:1 rectangle and measuring out 33.3° angles; but it's always more interesting to find a folding sequence that creates the exact reference marks needed. And after a bit of playing around with numbers and dimensions, I found a nice sequence that creates the edge unit from a 1:2 rectangle.



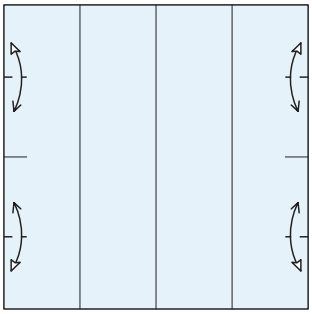
1. Begin with a square, white side up. Fold the paper in half from side to side and unfold.



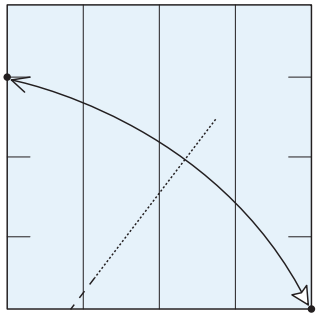
2. Fold the side edges in to the center, crease, and unfold.



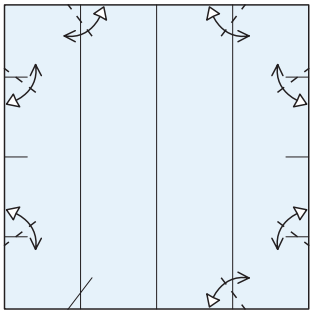
3. Fold the top down to the bottom, making a pinch along the sides, and unfold.



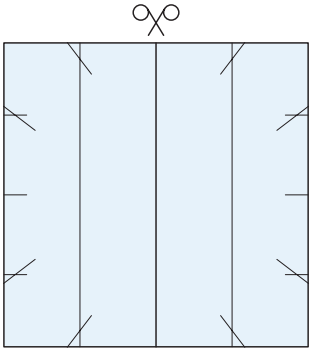
4. Fold the top edge down to the pinch you just made, making new pinches along the sides; unfold. Repeat with the bottom edge.



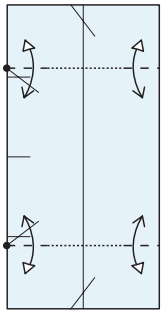
5. Bring the bottom right corner to the 3/4 mark along the left side; make a pinch along the bottom and unfold.



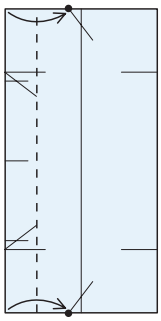
6. Repeat similarly for the other seven similar corner/pinch combinations.



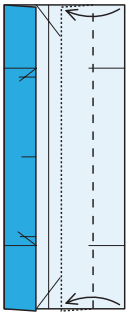
7. Cut the paper in half vertically down the center line. Put one of the pieces aside to make another unit.



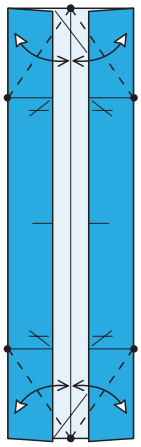
8. Fold the bottom edge upward using a pinch as a reference; make horizontal pinches along both the left and right edges. Repeat above.



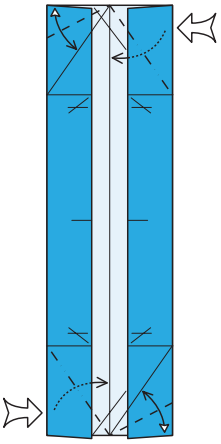
9. Fold the left edge inward so that the corners touch the pinch marks.



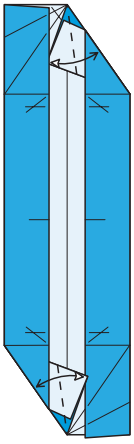
10. Fold the right edge inward so that it matches the left.



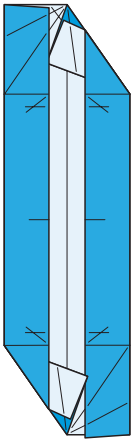
11. Fold the corners down and unfold. Note that the crease formed connects two reference points.



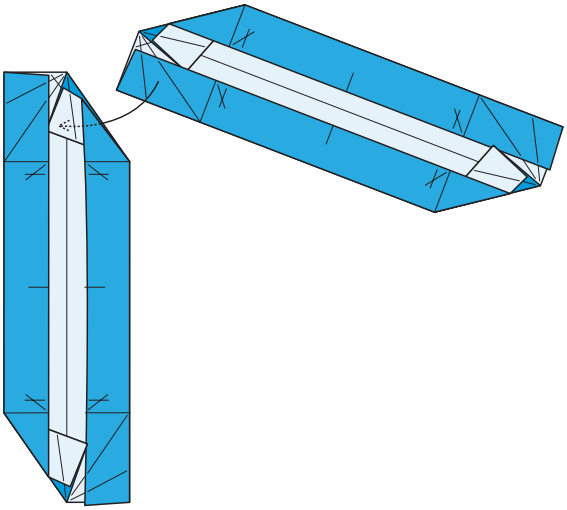
12. Fold the upper left corner to the angle bisector and unfold. Reverse-fold the upper right corner. Repeat below.



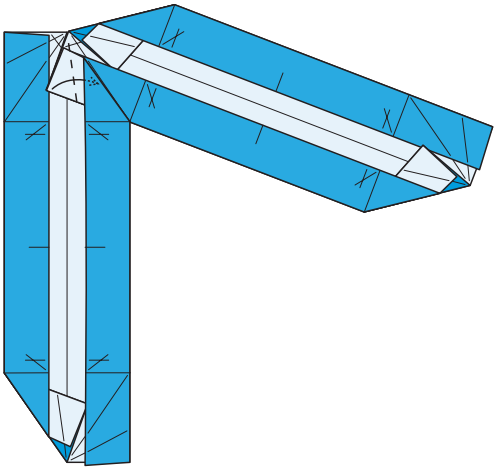
13. Fold and unfold along the angle bisector. Repeat below.



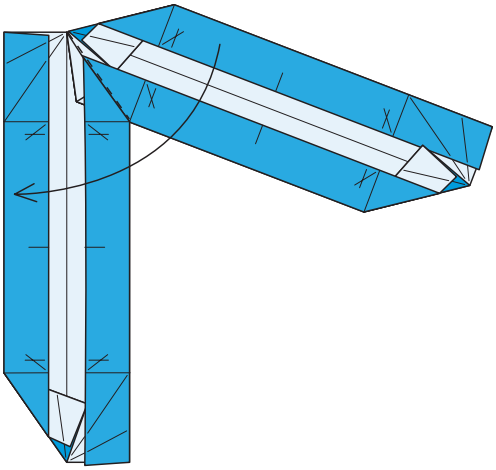
14. Finished unit. You can use the other half of the paper for another unit; fold a total of 12 units.



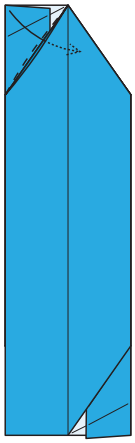
15. To join two units, tuck the corner of one unit into the pocket of another unit.



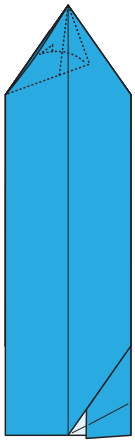
16. Fold the corner over on the existing crease and tuck it underneath the raw edge.



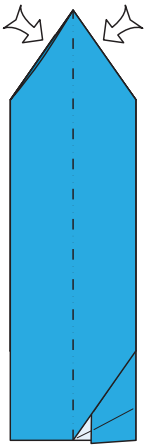
17. Fold the unit down so that it lies on top of the other.



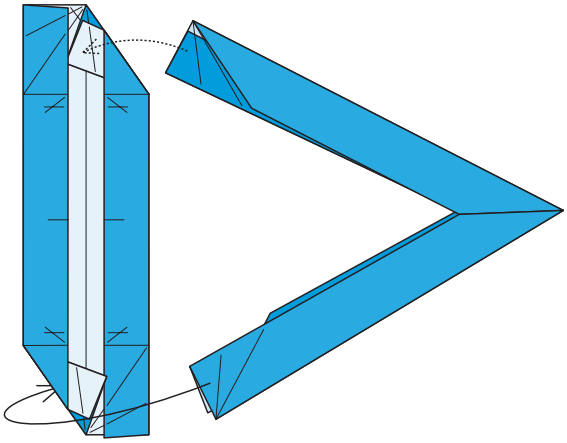
18. Tuck the flap into the pocket.



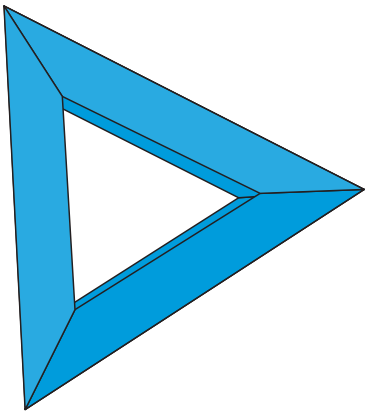
19. Fold the hidden corner over and tuck it under an edge as you did in step 16.



20. Squeeze the sides of the unit, making it three-dimensional.

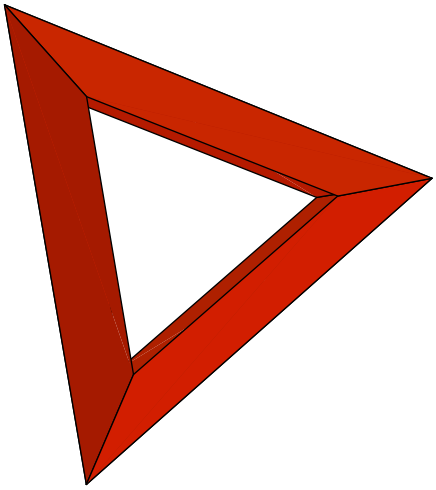


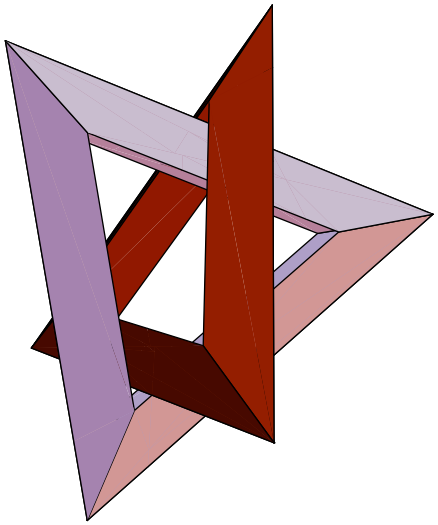
21. Attach a third unit in the same way to make a triangular ring.

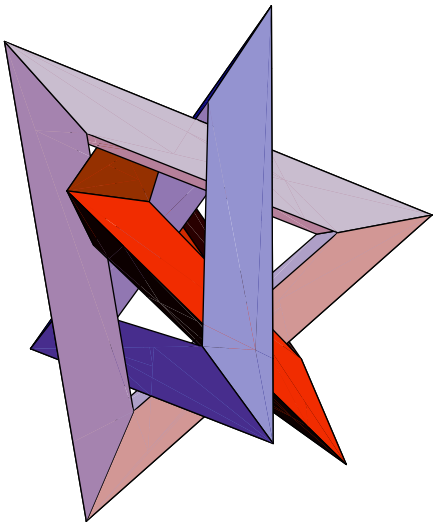


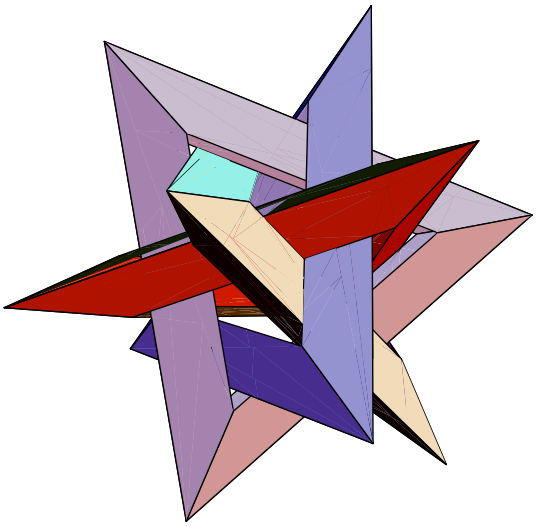
22. Build four rings and interlock them to form the polyhedron.

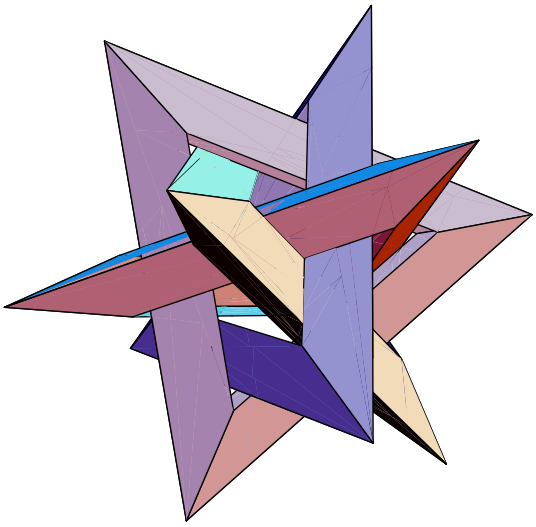
These can be assembled into the polypolyhedron the same way as was done for the 30-stick figure. Actually, it's considerably easier. The assembly sequence for the 12-stick model is shown below.





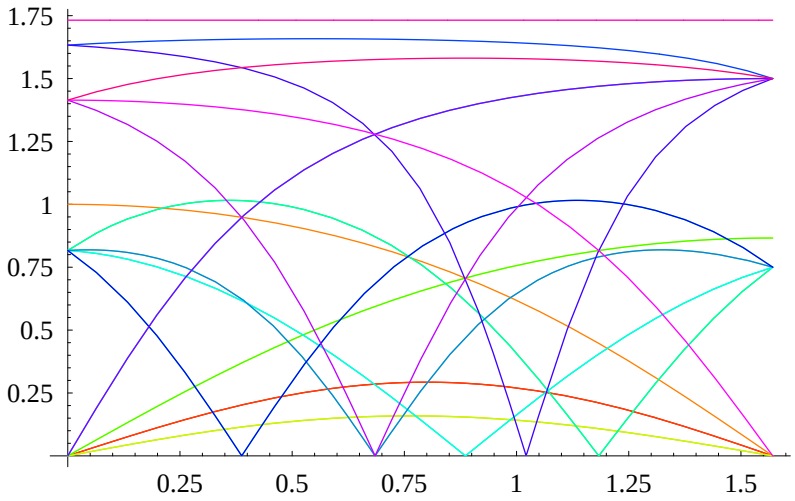






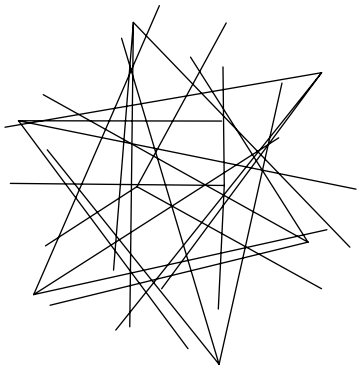
As in the compound of 10 triangles derived from the dodecahedron, you can alternatively make each triangular ring from a single strip of paper three times the length of a single unit.

The next stop was the Cuboctahedron, with 24 edges. Its distance plot is given below.

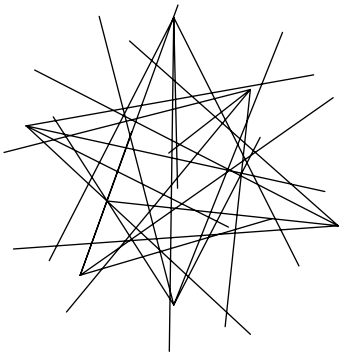


This looked much more fruitful than anything so far: there are five zero crossings between 0 radians (the original cuboctahedron) and $\pi/2$ radians (its dual, the rhombic dodecahedron). So, with eager anticipation, I plotted the skeletons of all five.

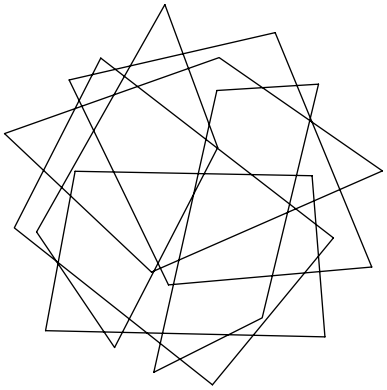
$\phi \to 0.387597$



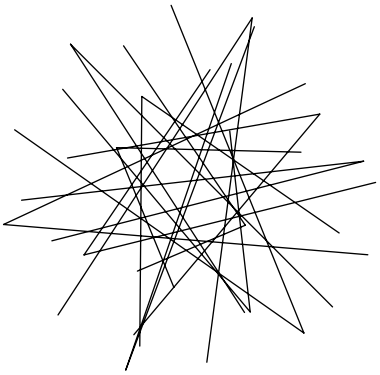
$\phi \to 0.684719$



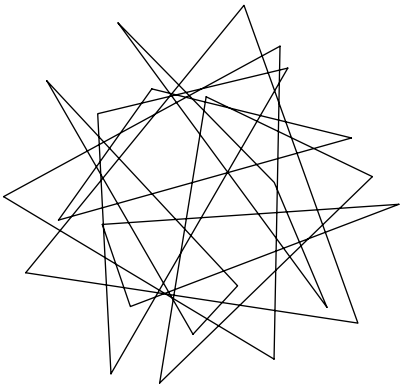
$\phi \to 0.886077$



$\phi \to 1.02133$



$\phi \to 1.1832$

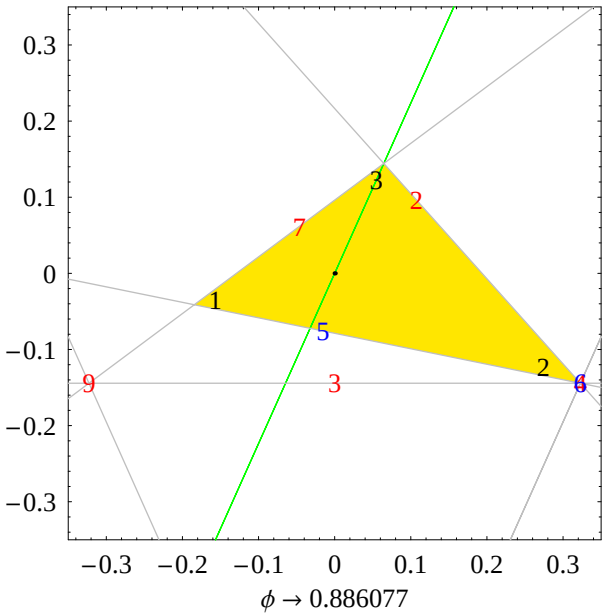


This brought something new. Although in each skeleton, all edges were joined to other edges, they weren't necessarily joined at *both ends*. For three of the roots — ϕ=.387597, ϕ=.684719, and ϕ=1.02133, the edges are joined in clusters of 2 or 3 edges — but at one end only.

Well, this was one step further away from the concept of polyhedra. These weren't even *polygons*. But I could still imagine making polypolyhedra out of them, except for one small problem. Remember the structural instability for structures made from unnotched sticks? If the sticks aren't joined at both ends, then the same structural instability exists. That would allow the polypolyhedron to collapse, although infinitesimally less quickly than burrs composed of entirely unjoined sticks. So in fact,

for the cuboctahedron, there are really only two stable skeletal solutions. For ϕ=0.866077, we have a 6×1×4 skeleton, and for ϕ=1.1832, we have an 8×1×3 skeleton.

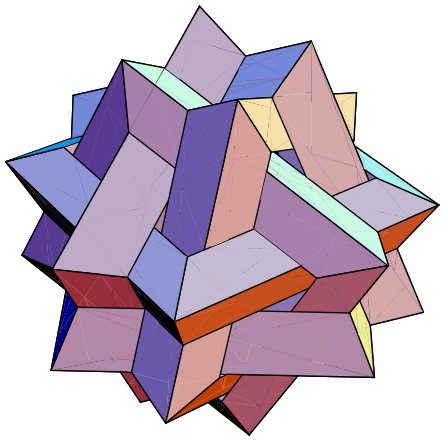
The 6×1×4 skelton has a cross section plot like this.



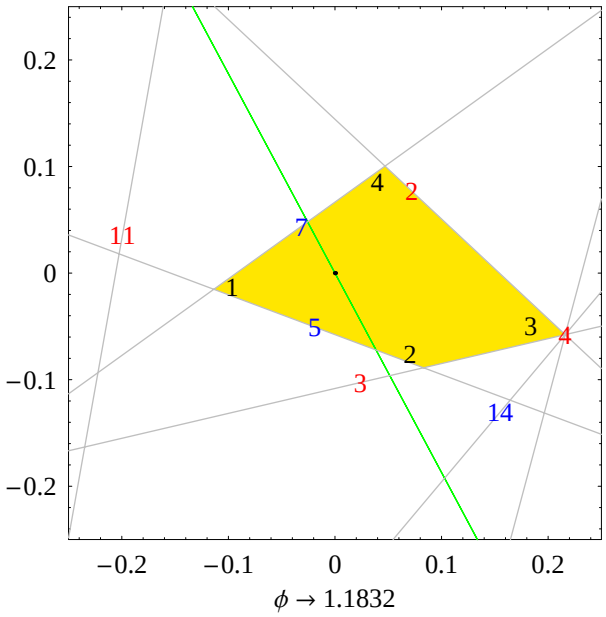
Turning this into an edge unit analogous to the previous ones, the unwrapped edge looks like this:

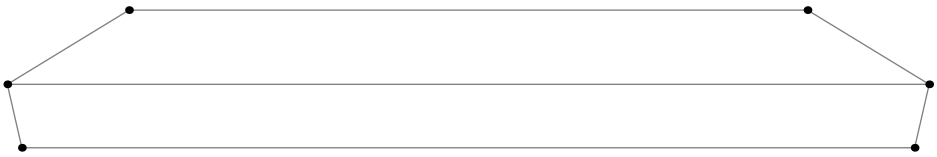
0.0833333	0.367788	0.916667	0.367788
0.	0.163663	1.	0.163663
0.142857	0	0.857143	0

And the resulting polypolyhedron is this.

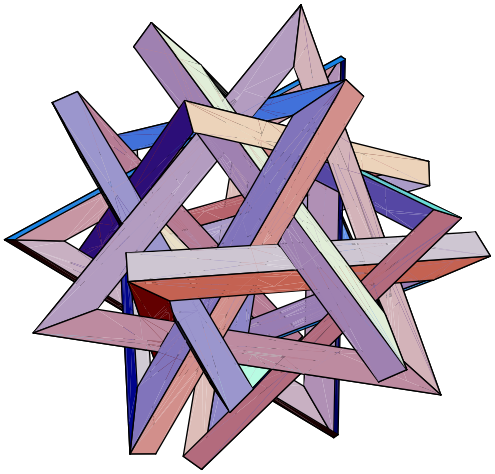


Similarly, for the 8×1×3 skeleton, the cross section, unwrapped stick, and polypolyhedron are shown below.

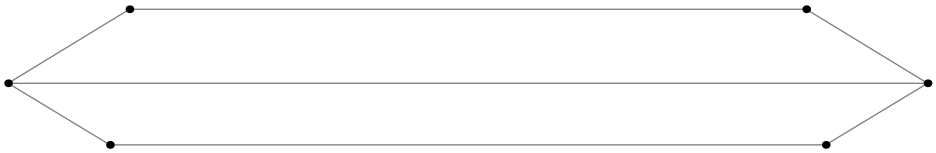




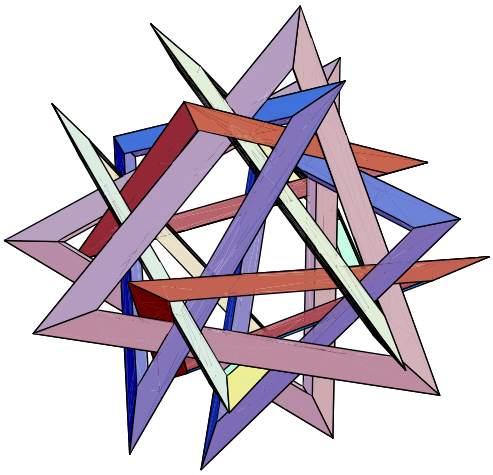
0.132	0.149323	0.868	0.149323
0.	0.0688247	1.	0.0688247
0.0157895	0	0.984211	0



This was the configuration that gave the widest possible stick whose cross section is given by vertices 1, 4, and 3, respectively, in the cross-section plot above. We can obtain sharper corners on the polyhedron by picking a different cross section for the stick. Using 2, 4, and 3 as the vertices gives the stick plan and polypolyhedron below.



0.132	0.147488	0.868	0.147488
0.	0.0669894	1.	0.0669894
0.110769	0	0.889231	0

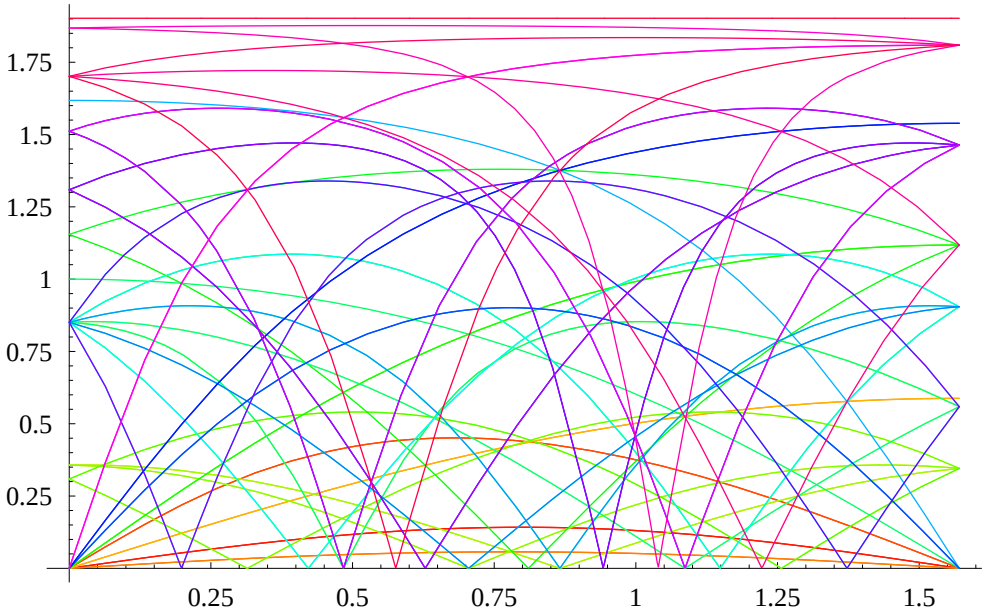


At this point, the score was: Tetrahedron, nothing; Octahedron, one polypolyhedron, the 4×1×3; Cuboctahedron, two polypolyhedra, the 6×1×4 and 8×1×3; and Dodecahedron, three polypolyhedra, the 10×1×3, the 5×4×3, and the 6×1×5.

Since we're pausing here, I should point out that Michael Naughton had previously discovered and utilized the 6×1×4 structure for his own polypolyhedron, composed of 6 square rings. But, as I said before, I'm not put off if someone else has picked up some of the shiny stones already. As Bill Gates reportedly said about Steve Job's raid on Xerox PARC, "Just because Steve stole their stereo doesn't mean I can't come in and take their TV, too."

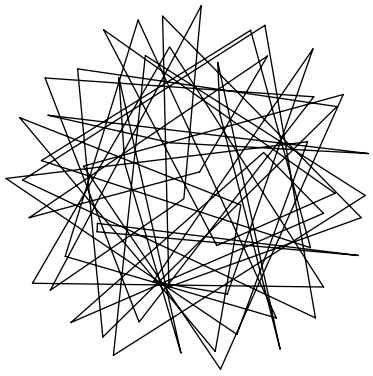
There was one uniform-edge structure left: the Icosidodecahedron. I had high hopes for this polyhedron as a starting point. Its 60 edges, twice as many as the dodecahedron, seemed sure to offer rich and exciting opportunities.

First came the plot of distances. The distance between a reference edge and each of the 59 other edges is plotted below as a function of edge rotation angle.

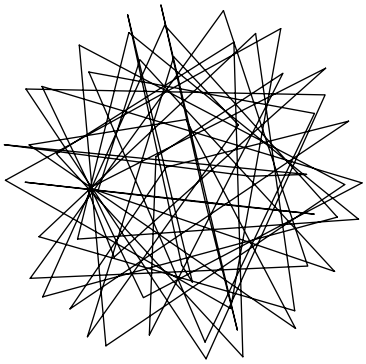


This looked very promising indeed. There are, if you count carefully, 16 distinct zero crossings (not including 0 radians, the original icosidodecahedron, and $\pi/2$, its dual, the rhombic triacontahedron). But how many of these would correspond to sticks joined at both ends? Just six, as it turned out. Their skeletons are shown below.

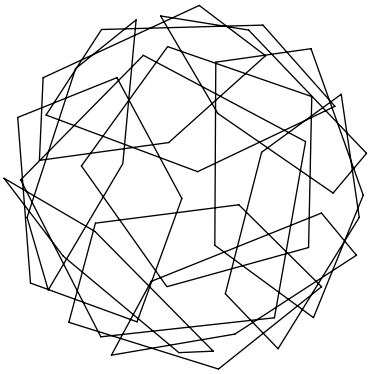
$\phi \to 0.422045$



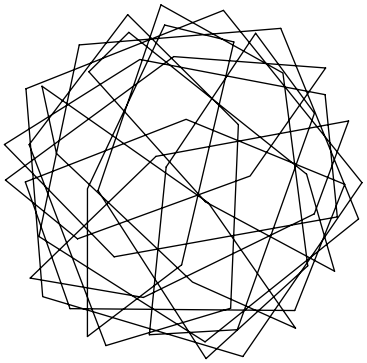
$\phi \to 0.628319$



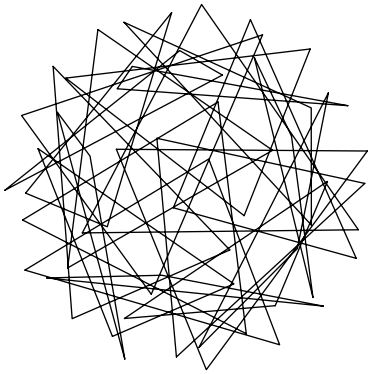
$\phi \to 0.865925$



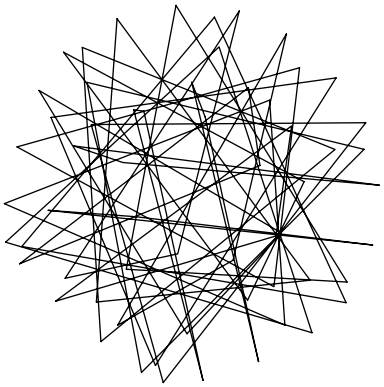
$\phi \to 1.08678$



$\phi \to 1.25664$



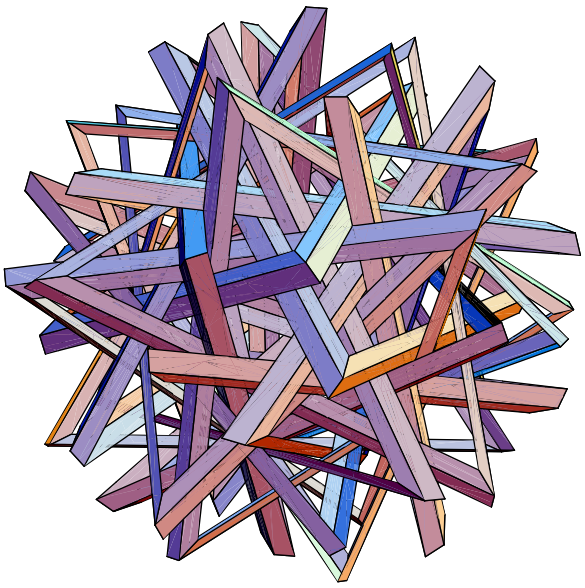
$\phi \to 1.37262$

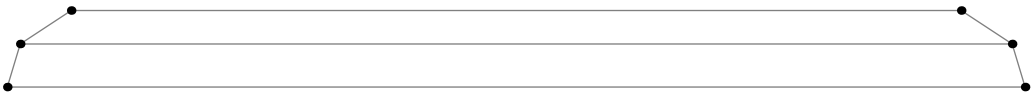


This offered another surprise: although there are six distinct polypolyhedral skeletons derived from the icosidodecahedron, they are of only two basic types. The roots ϕ=.865925 and ϕ=1.08678 are both 12×1×5, which the other 4 are all 20×1×3. But although they may have the same classification, they are distinctly different; the pentagons and triangles loop through each other in different ways, and so there are six distinct polypolyhedra to be found.

These stick figures turned out to be incredibly complex, and I confess that I have not yet had the fortitude to construct any. But they are intriguing to examine; and so I will show the stick patterns and the finished models. And perhaps, some intrepid soul seeking an origami modular challenge, will endeavor to construct one or two of them.

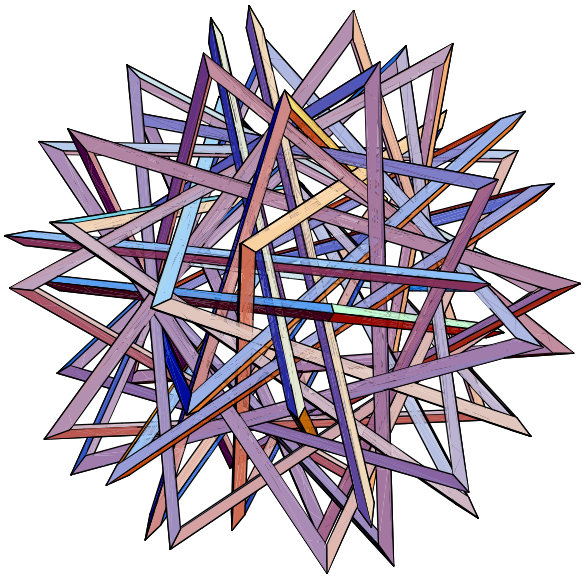
First, a 20×1×3, polypolyhedron and stick pattern.





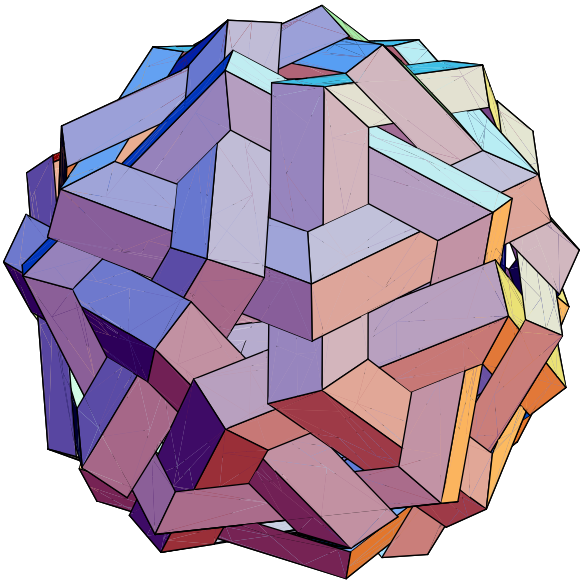
0.0627725	0.0751854	0.937227	0.0751854
0.0127118	0.042302	0.987288	0.042302
0.	0	1.	0

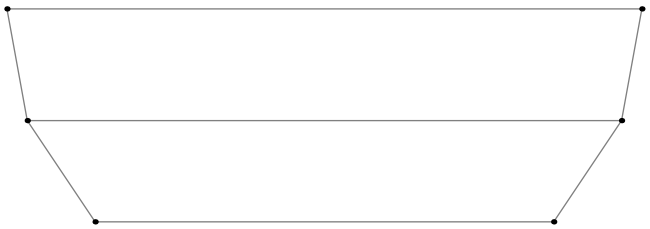
Next comes the polypolyhedron and its stick pattern for the second root, also a 20×1×3 configuration.



0.0349016	0.0539716	0.965098	0.0539716
0.	0.0290571	1.	0.0290571
0.0428911	0	0.957109	0

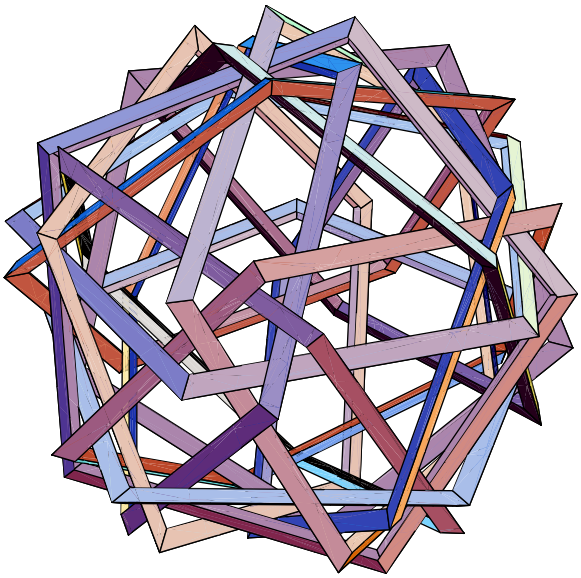
The next polypolyhedron is made from a 12×1×5 arrangement. It makes a nice woven ball.

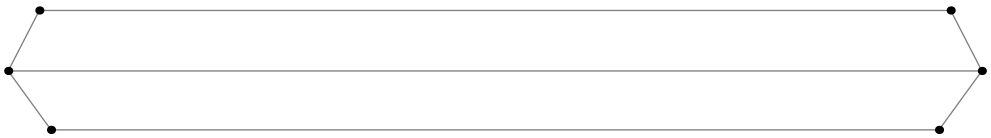




0.	0.335361	1.	0.335361
0.0320063	0.159436	0.967994	0.159436
0.138877	0	0.861123	0

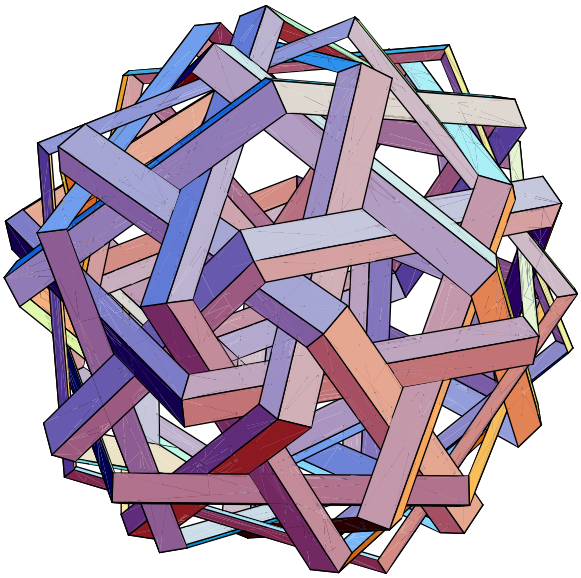
The fourth arrangment goes back to a 20×1×3 configuration.

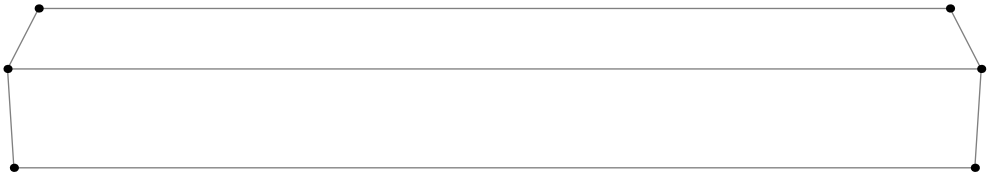




0.0320149	0.122659	0.967985	0.122659
0.	0.0605452	1.	0.0605452
0.0439887	0	0.956011	0

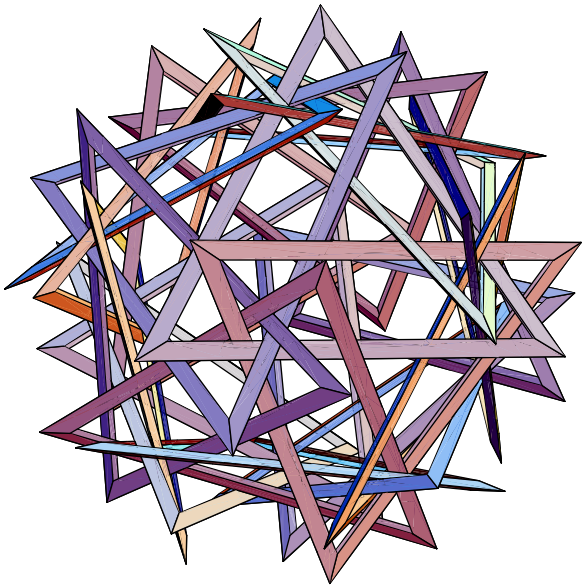
By selecting different planes of the cross section to define the sticks, we can construct a more woven-looking polypolyhedron with the same structure but with different stick dimensions.





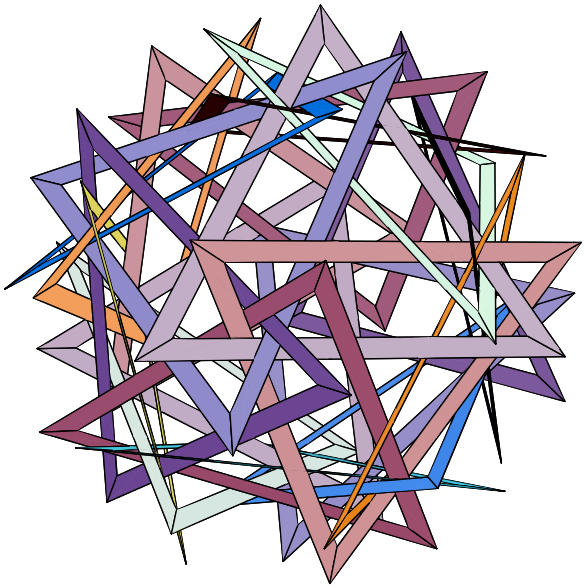
0.0320149	0.163648	0.967985	0.163648
0.	0.101535	1.	0.101535
0.00653032	0	0.99347	0

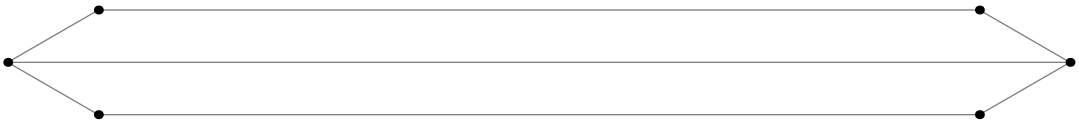
The next root is a 20×1×3 configuration, but the triangles are arranged differently than the previous root.



0.0715683	0.0786116	0.928432	0.0786116
0.	0.0364568	1.	0.0364568
0.0567265	0	0.943273	0

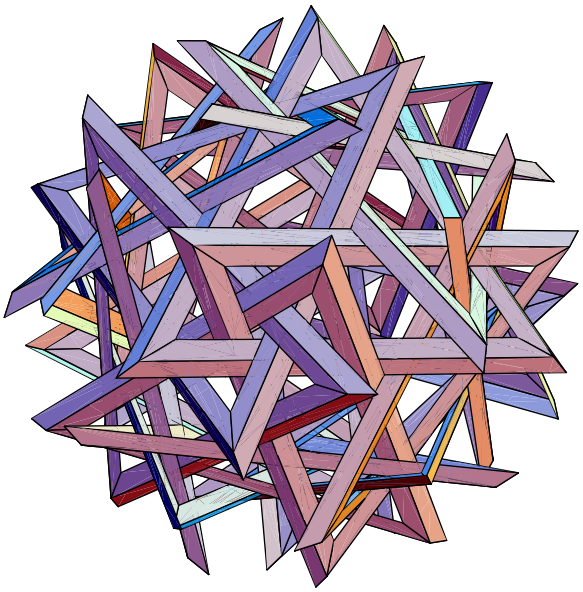
The stick does not necessarily have to be three-dimensional; in the next polypolyhedron, which is made from the same pattern, the sides of the stick are folded flat but they still lock each other together. You will have to make this stick from very stiff paper, however, to avoid it bending from side to side.





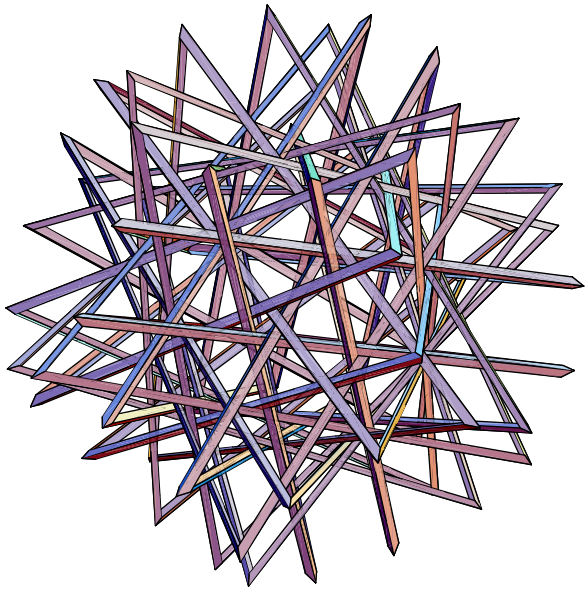
0.0853026	0.0985166	0.914697	0.0985166
0.	0.0492583	1.	0.0492583
0.0853026	0	0.914697	0

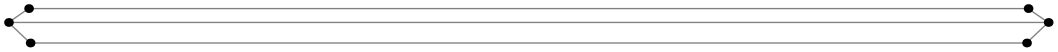
Yet another selection of facial planes in the cross section gives still another polypolyhedron. I particularly like this one because of the prominent pentagrams formed by the overlapping corners of each face.



0.119745	0.0903186	0.880255	0.0903186
0.0415267	0.0451514	0.958473	0.0451514
0.	0	1.	0

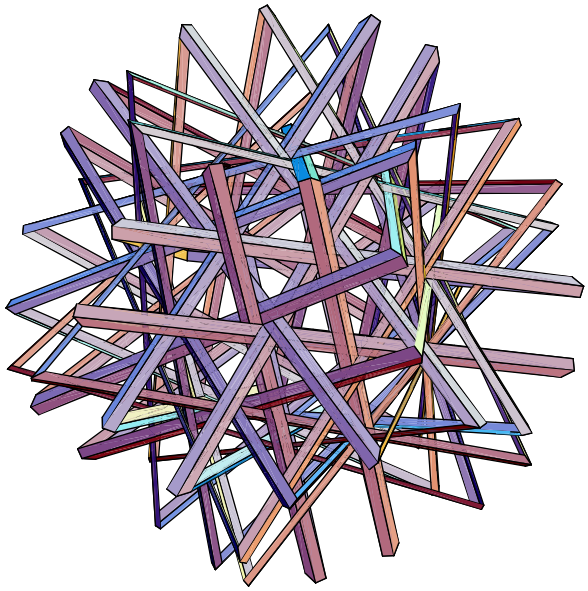
The last configuration gives the thinnest sticks of all — perhaps an aesthetic deficiency because it makes it difficult for the folded model to hold its shape, but it results in a delicate and lacy polypolyhedron.





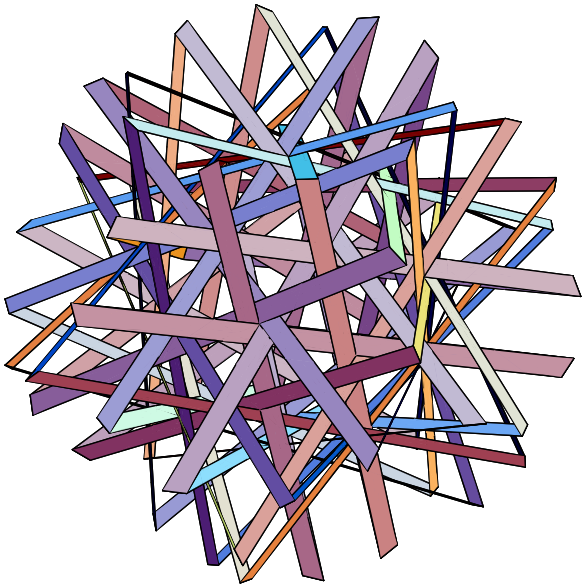
0.0193283	0.0331042	0.980672	0.0331042
0.	0.0197548	1.	0.0197548
0.0208262	0	0.979174	0

A different selection of cross section facial planes gives a slightly wider stick.



0.00390855	0.0427891	0.996091	0.0427891
0.	0.0197548	1.	0.0197548
0.0208262	0	0.979174	0

An even wider stick can be found if we allow the stick to have a zero dihedral angle, i.e., the two halves of the stick are coplanar.



0.017051	0.0802558	0.982949	0.0802558
0.	0.0401279	1.	0.0401279
0.017051	0	0.982949	0

Some closing thoughts. I approached this problem from the standpoint of performing transformations on the edges of existing polyhedra. In doing so, I found groups of edges that formed polygons (or in only one case — FIT — polyhedra). The individual edges bore relationships to each other through symmetry operations; the polygons, as well, were related through other symmetry operations. All of the symmetry operations were finite rotations in three dimensions. The study of the relationships among operation such as rotations is part of the mathematical field of group theory; and as it turns out, all of the patterns I found can be derived from subgroups of three finite rotation groups, corresponding to tetrahedral, octahedral, and icosahedral symmetries.

Having now worked my way through all of the convex polyhedra and having derived many polypolyhedra suitable for origami implementation, it was appropriate to pose the question: had I found them all? And the answer, after some reflection, turned out to be: not by a long shot! In fact, there were whole classes of polypolyhedra that I had missed. The most glaring omission was that none of my polypolyhedra were 2-uniform-vertex. What's this? Well, I had required that my polypolyhedra be uniform-edge (all their edges are alike). But there are two types of uniform-edge polypolyhedra: those in which the

vertices are all alike — and I had found a lot of those — and those in which the vertices are of two different types (in which case, each edge has one vertex of each type). If there are two types of vertex, the polyhedron is said to be 2-uniform-vertex. The rhombic dodecahedron is 2-uniform-vertex — some of the vertices are 3-valent and some are 4-valent — but I hadn't found any 2-uniform-vertex polypolyhedra, and I knew there had to be some. For example, take any one of the 1-uniform-edge, 1-uniform-vertex polypolyhedra I've constructed so far and break each edge into two identical halves: it becomes 2-uniform-vertex. There's obviously an entire family of 2-uniform-vertex polypolyhedra out there. The worrisome thing was that my systematic procedure had found none of them.

In fact, the rotating-edge algorithm that I'd used to construct polypolyhedra, while obviously fruitful, carried no guarantee of completeness. But what if there was something really, really interesting out there that wasn't constructible by rotating edges? I needed to find a construction algorithm that would provide such a guarantee, and it wouldn't be based on rotating edges.

The key to obtaining this guarantee would lie in the symmetries that a uniform-edge polypolyhedron must possess. To find all such polypolyhedra, I must first understand the possibles symmetries of such a structure. That led me down an entirely different avenue of investigation of polypolyhedra. And it turns out that there are a host of other 1-uniform-edge polyhedra out there, including both 1-uniform-vertex and 2-uniform-vertex varieties. Most of them share a property with the ones found thus far: they have at least some divalent vertices. But there is just one more polypolyhedron that has vertex valency greater than two — a property shared only with Five Intersecting Tetrahedra. I'll leave its identity as a puzzle for the reader. The solution of that puzzle, plus an enumeration of all interesting polypolyhedra, will be the topic of my next installment.

Robert J. Lang, January, 2000