Origami Burrs and Polypolyhedra, Part III

My search for modular origami polypolyhedra had led me on a meandering path through the uniform and semiuniform polyhedra that turned up a variety of interesting structures but there was the nagging question: what had I missed? And I knew I had missed something: my design scheme had found no 2-uniform-vertex polypolyhedra. So I resolved to develop a systematic search that would turn up all possible uniform-edge polypolyhedra that could be turned into origami polypolyhedra.

The standard references on polyhedra -- Coxeter, Wenninger, and so forth, turned up no leads. After a bit of research and some idle Web-surfing, I ran across the key concept in a newsgroup posting by John Conway on the existence of symmetric hecatohedra (polyhedra with 100 faces); Conway pointed out that you could classify all such polyhedra by their symmetry properties in a way that let you quickly enumerate all possible configurations. Aha! Symmetry — that was the answer. I could identify and enumerate all possible polypolyhedra by dividing them up according to their symmetry.

To start, I needed a precise definition of a polypolyhedron. Here it is:

A polypolyhedron is a compound of multiple linked polyhedra with uniform nonintersecting edges.

Let's run through the conditions:

It is composed of *multiple* polyhedra — there must be more than one. Of course, all of the polyhedra must be identical.

They must be topologically *linked* to one another, i.e., if they were made of string, you couldn't take them apart without cutting the string.

The edges are *uniform*, meaning every edge is like every other edge, both in length, and in how other edges are connected to its vertices. However, although all edges must be uniform — meaning there's only one type of edge — the vertices could be 1-uniform or 2-uniform. The vertices are 1-uniform if they are all alike. They are 2-uniform if there are 2 distinct types of vertices. If the vertices are 2-uniform, it's easy to see that every edge must have one vertex of one type and one of the other.

The edges are *nonintersecting*, meaning no edge touches any other edge other than at its endpoints.

We can make a few observations immediately. One is that the vertices must be at least 2-valent (at least 2 edges meet at every vertex). If any vertex is 1-valent, then the attached edge can't be part of a loop, but loops are the only way that the constituent polyhedra can be linked.

Note that nowhere do we stipulate that the facial polygons of the constituent polyhedra must be planar. We will, in fact, find many polypolyhedra whose facial polygons are skew (nonplanar).

Now, the uniformity of the edges means that the polypolyhedron must have certain rotational symmetries. If any two distinct edges of a polypolyhedron are identical, then it must be possible to rotate the polypolyhedron so that when one edge is brought into coincidence with the original position of any other edge, the polypolyhedron must be unchanged by the rotation. An object that is unchanged by rotation is said to have rotational symmetry.

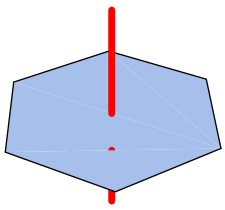
A polypolyhedron, therefore, can be classified according to its rotational symmetry.

If we arbitrarily pick a single edge of the polypolyhedron to be the "reference edge," then every other edge can be obtained from the reference edge by applying a rotational transformation. (The reference edge can be obtained from itself by applying the identity transformation.) Therefore, there is a one-to-one correspondence between each of the edges of a polypolyhedron and some rotational transformation.

Let's consider, for moment, the properties of the set of rotations that leave a polypolyhedron unchanged. First, there is one "rotation" where you don't actually do anything — called the *identity* rotation. Second, for any rotation, there is another rotation that returns the polypolyhedron to its original configuration — in other words, there is an *inverse* for every rotation. Third, if you follow one such rotation by a second, the polypolyhedron is still left unchanged, which means that the combination of those two rotations is still part of the set and could be expressed as a single rotation. This is a property called *closure*. Fourth, there is only a *finite* number of such rotations since there is only a finite number of edges. The combination of the first three properties — identiry, inverse, and closure — says that the set of rotations of the edges of a polypolyhedron form a mathematical entity called a *group*. And since the number of rotations (called the *order* of the group) is finite, they form a *finite group*. And last, since the elements of the group are 3-D rotations, they are called a *finite rotation group*.

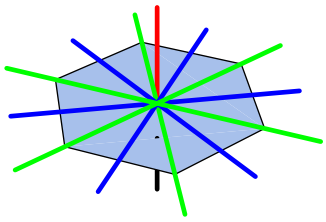
This observation lets us divide and conquer the problem of enumerating all polypolyhedra. The rotational symmetries of a polypolyhedron must form a finite rotation group — but there is only a small number of finite rotation groups to choose from. There are 5 families of rotation groups in 3-D, called the cyclic, dihedral, tetrahedral, octahedral, and icosahedral groups. The rotation group of any polypolyhedron must fall into one of these 5 families. Therefore, let's spend a moment discussing each of these symmetry groups.

The cyclic rotational group of order n, called C_n, is the set of rotations that leave a polygon of n sides unchanged. There's a different group for every value of $n \geq 2$, so really, there is a family of rotational groups. The order of the group — the number of distinct rotations — is simply n. The figure below shows the $n=6$ polygon, a hexagon.



The red line through the center of the polygon and perpendicular to it has an interesting property; any point on the line is left unchanged by any rotation in the group. Such a line with the property that it is left unchanged by rotations is called an *axis of rotation*. Each cyclic group has exactly one axis of rotation.

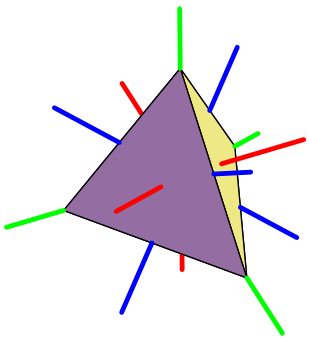
The next group is the dihedral group, D_n, which, like the cyclic group, is a family of groups, one for each n. The dihedral group D_n consists of the rotations that leave a polygon of n sides unchanged — but you're also allowed to turn the polygon over, which is a twofold rotation. The dihedral group D_n has $2n$ elements (n rotations about the vertical axis and n combinations of rotate-plus-turn-over). For n even, it has $(2n+1)$ axes of rotation: 1 vertically through the center of the polygon (shown in red), n through each vertex (shown in green), and n through the midpoints of the edges (shown in blue).



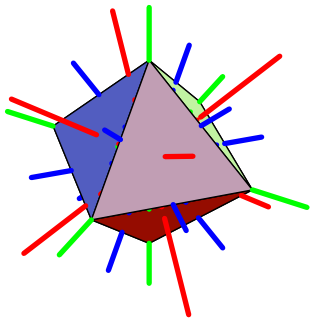
For n odd, the number of rotational axes is just $(n+1)$ because the green and blue axes turn out to be the same.

The elements of the cyclic group are also members of the dihedral group; the cyclic group C_n is therefore said to be a *subgroup* of the dihedral group D_n.

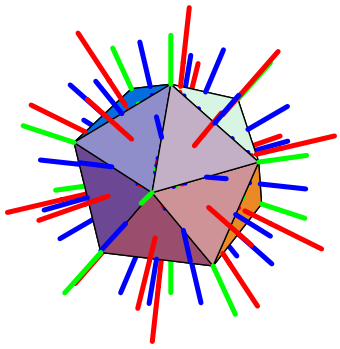
The next group is the tetrahedral group A_4, which consists of the rotational symmetries of a tetrahedron. The tetrahedral group is of order 12, which means that there are 12 distinct rotations (including the identity rotation) that leave a tetrahedron unchanged. You can see this by picturing the manipulations of a tetrahedron that leave it unchanged — there are 4 faces, and each face can have 3 different orientations, and 4×3=12. There are six axes of rotation through the midpoint of each edge, and another 4 that run through the midpoints of the faces and the opposite corners (red and green). Note that for the tetrahedron, the axes through the faces are collinear with the axes through the vertices.



The next group is the octahedral group, O_h, which consists of the rotational symmetries of an octahedron. The octahedral group is of order 24 (8 faces × 3 orientations per face). There are three distinct sets of axes of rotation: through the midpoint of each face (8, red), through the vertices (6, green), and through the midpoints of the edges (12, blue).



Last comes the icosahedral group A_5, the group of rotational symmetries of an icosahedron. The icosahedral group is of order 60 (20 faces × 3 orientations per face). The axes of rotation run through the midpoint of each face (20, red), through the vertices (12, green), and through the midpoints of the edges (30, blue).



You'll notice that for the dihedral, tetrahedral, octahedral, and icosahedral groups, the axes of rotation fall naturally into three sets, associated with the faces, edges, and vertices of the eponymous polyhedron.

You might wonder: the rotation groups are named after 3 of the 5 Platonic solids. What about the other two solids, the hexahedron (cube) and dodecahedron? They're already taken care of: the former has octahedral symmetry; the latter has dodecahedral symmetry. Every other polyhedron with rotational symmetry has symmetries that fall into one of these groups.

And so does every polypolyhedron. Since the 3D finite rotation groups are known, any polypolyhedron has to have the symmetry of one of these groups and therefore we can label every polypolyhedron by its rotation group. I use a single-letter label for the rotation group: Z,D,T, O, or I, for Cyclic, Dihedral, Tetrahedral, Octahedral, or Icosahedral symmetry. (Z is for Cyclic because I'm going to use C for something else later on.)

Now, a little thought experiment. Suppose I apply a rotational operator to a point. You get another point, which may or may not be distinct from the original point. The set of all distinct points obtained by applying all rotation operators in a rotation group to a single point is called an *orbit* of the point. The orbit is said to be *generated* by the point and the rotation group. Some of the time, the number of points in an orbit — which is called the order of the orbit — is equal to the number of rotations in the group. That is, for a given starting point — which we will call a seed point — there is a unique orbit point for every rotation operator in the group and vice-versa.

But not always. Any point that is on an axis of rotation is left unchanged by rotations about that axis. The orbits of such points are of lower order than the orbit of an arbitrary point. For example, for a tetrahedron (order=12), the vertices of the tetrahedron form an orbit of order 4. The midpoints of the faces form another orbit of order 4. And the midpoints of the edges form an orbit of order 6. Any point not lying on one of these lines is part of an orbit of order 12. Therefore, the tetrahedral group has several types of orbits with different numbers of points in the orbit. The same goes for the other rotational groups.

Let me give names to the three types of lower-order orbits. An orbit generated by a vertex of the eponymous polyhedron will be called a V orbit. An orbit generated by a midpoint of an edge will be called an E orbit; and an orbit generated by a midpoint of a face is an F orbit. Any other orbit will be called a C (for "complete") orbit. If a polypolyhedron has rotational symmetry, then any vertex of the polypolyhedron must lie in a V, E, F, or C orbit. The orders of the orbits of the various rotation groups are given in the following table.

	V	E	F	C
Cyclic(n)	—	—	1	n
Dihedral(n)	n	n	1	2 n
Tetrahedral	4	6	4	12
Octahedral	6	12	8	24
Icosahedral	12	30	20	60

Orbits matter because if you select two vertices of any single edge, each vertex generates an orbit. The vertices of every other edge must lie within those two orbits (which may or may not be the same). If we pick a single edge (the "seed edge") of the polypolyhedron, one of its vertices defines an orbit; every other edge must have one of its vertices in that same orbit. Similarly, the other vertex of the seed edge defines a second orbit (that may or may not be the same orbit as the first); every other edge must have its other vertex in that same orbit. Therefore, we can further classify uniform-edge polypolyhedra by the orbits of their vertices in addition to their rotational symmetry group.

In fact, we can fully characterize a uniform-edge polypolyhedron by its rotation group and a single edge, since all other edges may be obtained by application of the rotation operators of the symmetry group to that edge. We'll call that edge the "seed edge." Since an edge has at most two vertices, the vertices of a polypolyhedron taken collectively form at most two distinct orbits.

There are only four possible types of orbits for each vertex — V, E, F, or C. We'll label the seed edge with a symbol {*o1*,*o2*} according to the orbits of the two vertices of that edge. We'll call this symbol the *orbital type* of the edge — and by extension, of the polypolyhedron. For example, any edge of a {V,C} uniform-edge polyhedron has one vertex in a V orbit and the other in a C orbit. For a {*o1*,*o2*} polypolyhedron, there are 4 possibilities for *o1* and 4 possibilities for *o2*, so there are 16 possible orbital types — one for each possible pair of the four symbols V, E, F, and C.

In addition to these, there are two distinctly different types of double-letter combinations: a {V,V} edge could have both vertices as part of the same V orbit or part of two different V orbits. (Two V orbits are different if the orbits lie at different radial distances from the origin.) We can distinguish between these cases by appending a number to the letter: {V1,V1} denotes both vertices in the same orbit, {V1,V2} means they lie in different orbits. Similarly, we'll have both {E1,E1} and {E1,E2}, {F1,F1} and {F1,F2}, and {C1,C1} and {C2,C2}. So, in principle, there are a total of 20 different orbital types to consider.

Not all 20 types are distinct, however. There is no distinction between a {V,C} edge and a {C,V} edge other than the order in which you take the vertices for the seed edge. So we can eliminate some pairs as being redundant. After eliminating redundant types, we are left with 14 orbital types for each symmetry group: {V1,V1}, {V1,V2}, {V1,E1}, {V1,F1}, {V1,C1}, {E1,E1}, {E1,E2}, {E1,F1}, {E1,C1}, {F1,F1}, {F1,F2}, {F1,C1}, {C1,C1}, and {C1,C2}. Any polypolyhedron must be of one of these types. (It is possible for a polypolyhedron to fall into more than one, since the V and F orbits of the tetrahedral group are equivalent to each other under a rotation.) But these are all there are, so if I could enumerate all of the polypolyhedra of each orbital type, I could be assured that I had found *all* possible polypolyhedra.

Now that looks like there are a lot of polypolyhedra: there are 3 finite rotation groups (*T*,*O*,*I*) and two infinite groups (Z_n, D_n). For each rotation group, there are 14 possible types of edges; and within each orbital type, there is a large number of distinct edges (since I have a choice of many points within each orbit to use to construct the seed edge).

However, I can eliminate some of the configurations as being uninteresting right off the bat by considering the valency of the vertices of a polypolyhedron. For a rotational group of order *N*, the polypolyhedron has at most *N* edges (it could have fewer if two different rotation operators on an edge result in the same edge). Let's denote the order of an orbit by *O(T)* where *T* is the type of orbit (V,E,F, or C). Each edge contributes 2 vertices; thus, there are 2*N* total edge-to-vertex connections. These must be divided up among the vertices. Now, consider an edge of type {*T*, C} where *T* is an orbit different from the C orbit. The *N* edges contribute *N* connections to *T*-type vertices and *N* connections to C-type vertices. Thus, the valency of any C vertex is *N*/*O*(C), which is 1, because *O*(C)=*N* by definition. But a polypolyhedron with 1-valent vertices is uninteresting — the "polyhedra" are simply disjoint line segments. So we can eliminate any combinations of the form {*T*,C} — except for {C,C}. The {C,C} case is that where both vertices are not just in the same types of orbits, but are actually in the *same* C orbit.

Consequently, there are now only 10 distinct types of edges that lead to interesting polypolyhedra: {V1,V1}, {V1,V2}, {V1,E1}, {V1,F1}, {E1,E1}, {E1,E2}, {E1,F1}, {F1,F1}, {F1,F2}, and {C,C}.

These types don't necessarily apply to all symmetry groups, by the way; the cyclic group only has an F axis of rotation — no V or E axes — and so only supports {F1,F1} and {C,C} orbits.

We'll define some more terms: a *homoörbital* polypolyhedron is one in which both vertices lie in the same orbit (the polyhedron is therefore 1-uniform-vertex); a *heteroörbital* polypolyhedron is one in which there are two distinct orbits (and the polyhedron is 2-uniform vertex). In a heteroörbital polypolyhedron, every edge has one vertex in one orbit and one in the other orbit. The homoörbital polypolyhedra have edges of type {V1,V1}, {E1,E1},{F1,F1}, and {C,C}; the heteroörbital polypolyhedra have edges of types {V1,V2}, {V1,E1}, {V1,F1}, {E1,E2}, {E1,F1}, and {F1,F2}.

Now, within a single orbit type, we must ask: how many polyhedra are there? Or put differently, how can we further classify polypolyhedra within a single orbit type?

Since an edge of the polyhedron connects a vertex in one orbit with a vertex in the other, it is tempting to guess that if the orbit type is heteroorbital, say of form {*o1*,*o2*}, then there are *O(o1)O(o2)* possible edges and therefore the same number of possible polypolyhedra. But in fact, there are far fewer.

To collect every possible edge within an orbit of type {*o1*,*o2*}, we need only pick a single vertex from the first orbit as the first vertex of our seed edge since every other vertex in the orbit is related by a simple rotation. So instead of there being *O(o1*) possibilities for the first vertex, we need only consider 1 possibility because all others are equivalent under some rotation of the entire polypolyhedron.

Now, what about the second vertex of the seed edge? In principle, we should consider each of the *O(o2)* points in the *o2* orbit as the other vertex of the seed edge. So there are a total of O(*o2*) seed edges. But if the first vertex was on an axis of rotation, then there are rotations that leave the first vertex unchanged but not the second, meaning that this list of O(*o2*) seed edges contains some pairs that are equivalent under rotation.

By considering all O(*o2*) possibilities and then eliminating duplicates (edges equivalent under rotation), you can be assured of finding every possible edge within the orbit type. And so, by considering all 10 orbit types for each symmetry group, and all 5 symmetry groups, we can identify every possible edge type and therefore, every possible polypolyhedron.

Each polypolyhedron can be labeled by four quantities: its symmetry group; the orbit of the first vertex in the seed edge; the orbit of the second vertex; and an indicator of which point in the second orbit is the second vertex. I chose a labeling system based on the original rotation operators.

For each symmetry group, the rotation operators are labeled g_i, $i = 1 .. N$, where N is the order of the group. Let $p^{(o2)}$ be some point in orbit *o2*; then $g_i \cdot p^{(o2)}$ is also a point in the orbit for every i. Conversely, for every point q in the orbit, I could find at least one operator g_i such that $q = g_i \cdot p^{(o2)}$ for a reference point $p^{(o2)}$. (There could be more than one; for V, E, and F orbits, a given g_i, and therefore i, is not unique, for a given q.) I could therefore label the second vertex by the index i. Thus, for symmetry group G having symmetry operators g_j, orbits O1 and O2, and second orbit index i, the two vertices of the seed edge would be given by $p^{(O1)}$ and $g_i \cdot p^{(O2)}$, respectively.

So, finally, I had a method to enumerate all possible polypolyhedra. For each symmetry group G∈{Z_n, D_n, T, O, I} and orbit types {O1,O2}∈{{V1,V1}, {V1,V2}, {V1,E1}, {V1,F1}, {E1,E1}, {E1,E2}, {E1,F1}, {F1,F1}, {F1,F2}, {C,C}}, I picked a point in the orbit $p^{(G,O1)}$ to be the "seed point" for that orbit. Then the seed edge is the line segment from $p^{(G,O1)}$ to $g_i \cdot p^{(G,O2)}$ for all possible values of i=1..O(O2).

I should note that this prescription gives some duplicates, since if O2 is not a C orbital there are by definition multiple rotation operators g_i that give the same second vertex $g_i \cdot p^{(G,O2)}$. But the important thing was that every possible edge seed (and hence every possible polypolyhedron) would be given by this prescription. By constructing all possible edge seeds and sorting out the duplicates, I could be assured of finding every possible polypolyhedron.

And so, that's what I did, using *Mathematica*™. I set up the operators for each of the rotation groups, constructed representative orbits, and then for each orbit type, constructed and sorted the unique edges within that orbit type.

And I could eliminate one symmetry group right off the bat — the cyclic group. All of the orbits of the cyclic group lie in a plane. If the polypolyhedron is homoörbital, all of the vertices lie in a plane which means the polypolyhedra are either intersecting or unlinked. A similar argument shows the impossibility of a heteroörbital cyclic polypolyhedron. That left only the dihedral and three polyhedral groups. I suspected that there was no dihedral polypolyhedron as well, so I picked a single dihedral group, D_6, to check. After constructing all possible edges of all possible groups and orbit types, I found that among the three polyhedral rotation groups and D_6, there are 188 distinct edges — no more. Each edge is potentially a seed for a polypolyhedron.

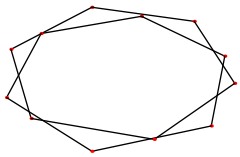
What next? Well, with each of those 188 edges, if you apply the rotation operators from the symmetry group, you obtain a set of edges that taken together potentially form a polypolyhedron. My *Mathematica* search had found 188 combinations of the four symmetry groups (T, O, I, and D), the ten valid and distinct orbit combinations ({V1,V1}, {V1,V2}, {V1,E1}, {V1,F1}, {E1,E1}, {E1,E2}, {E1,F1}, {F1,F1}, {F1,F2}, and {C,C}), and the assorted values of second orbit indices, which gave distinct seed edges. Because of rotational symmetry, applying a rotation operator g_j should give another edge of the polypolyhedron; applying all rotation operators should, and does, give all of the edges. And so, I now had a complete list of all possible polypolyhedra.

Well, not quite. What I'd done so far enforced rotational symmetry, uniform-edge, and 1- or 2-uniform vertices. However, I listed six conditions at the very beginning that were requirements for interesting polypolyhedra, and many of the 188 fail on one or more counts. Some of them were simple polyhedra, not compounds of multiple polyhedra. Some still have univalent vertices (I had eliminated combinations that were guaranteed to be univalent, but I hadn't done anything that absolutely insured multi-valency); some have intersections among the edges. All of these are no-no's, as far as interesting polypolyhedra are concerned. After constructing all 188 polypolyhedra, I used *Mathematica* again to weed out univalent or intersection-containing polyhedra, which left 41 polypolyhedra behind.

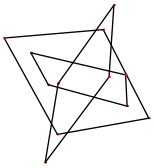
After computing the vertices and edges of the 41 polypolyhedra, I could also use *Mathematica* to compute some of the other properties, such as the number of polypolyhedra, the valencies of the edges, and the symbol that I introduced previously, P×N×M, where P is the number of polyhedra, N is the number of faces (polygons) per polyhedron, and M is the number of vertices (or edges) per polygon.

I also then picked a seed edge with the prescribed orbits and plotted the skeletons of all 41 polypolyhedra thus obtained. The labels for the plots are the symmetry group, vertex orbits, and orbit index for the second vertex; the valency of each edge ("2-3" means one vertex is 2-valent and the other is 3-valent), and the number of polyhedra, faces per polyhedra, and vertices (edges) per face.

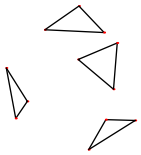
D C C 6, 2−2, 2×1×6



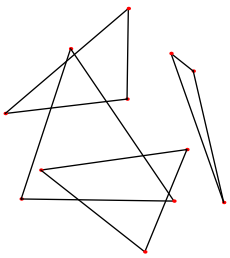
T E1 E2 3, 2−2, 3×1×4



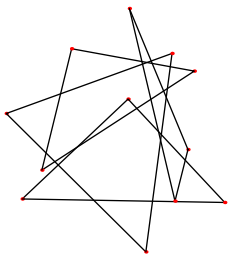
T C C 7, 2−2, 4×1×3



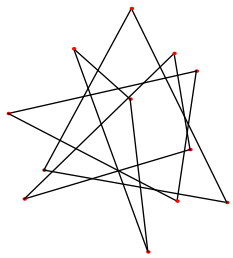
T C C 2, 2−2, 4×1×3



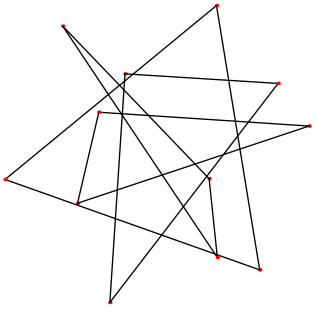
T C C 10, 2−2, 4×1×3



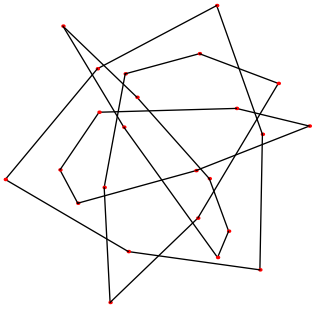
T C C 12, 2−2, 4×1×3



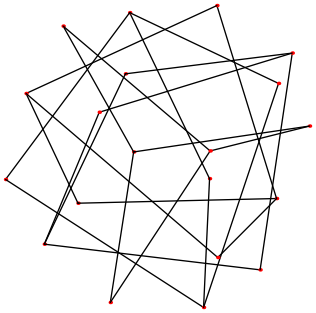
O E1 E1 5, 2−2, 4×1×3



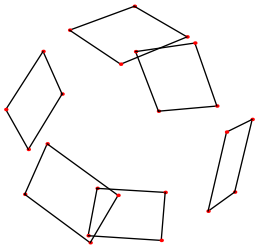
O E1 E2 2, 2−2, 4×1×6



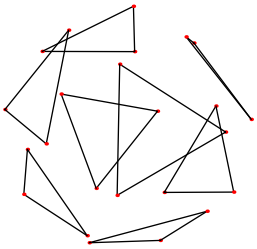
O E1 F1 4, 2−3, 4×3×4



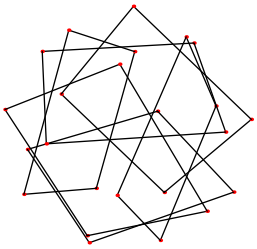
O C C 4, 2−2, 6×1×4



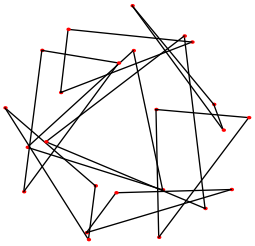
O C C 2, 2−2, 8×1×3



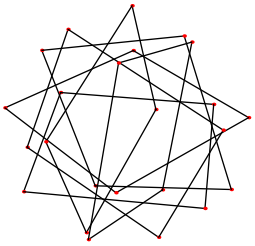
O C C 11, 2−2, 6×1×4



O C C 9, 2−2, 8×1×3



O C C 6, 2−2, 6×1×4



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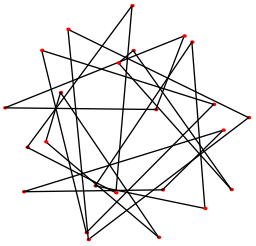
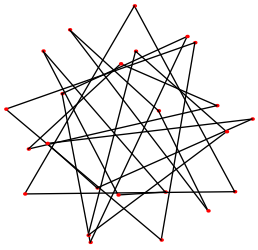
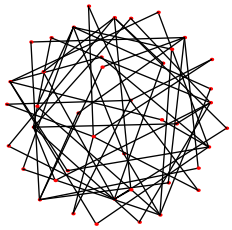


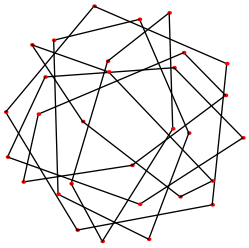
O C C 14, 2−2, 8×1×3



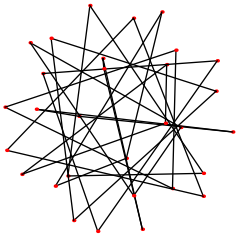
I V1 E1 4, 5−2, 6×5×4



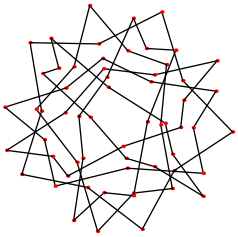
I E1 E1 5, 2−2, 6×1×5



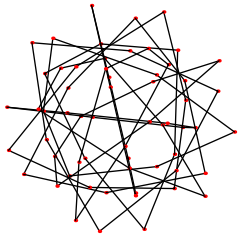
I E1 E1 24, 2−2, 10×1×3



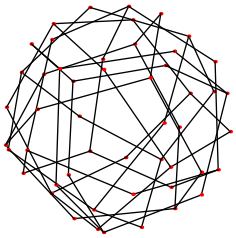
I E1 E2 6, 2−2, 6×1×10



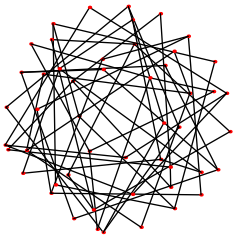
I E1 E2 20, 2−2, 10×1×6



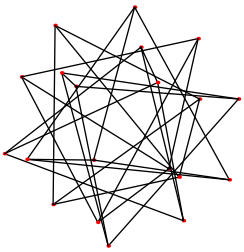
I E1 F1 6, 2−3, 5×4×6



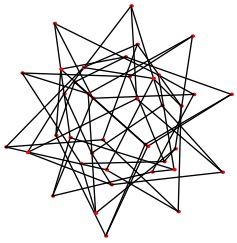
I E1 F1 7, 2−3, 10×3×4



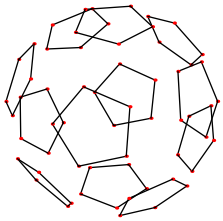
I F1 F1 23, 3−3, 5×4×3



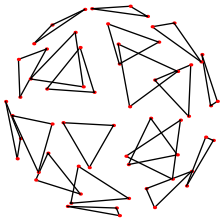
I F1 F2 7, 3−3, 5×6×4



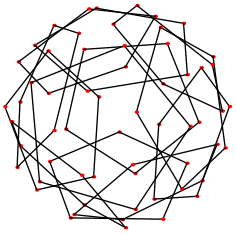
I C C 14, 2−2, 12×1×5



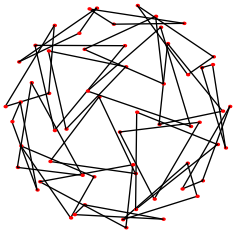
I C C 3, 2−2, 20×1×3



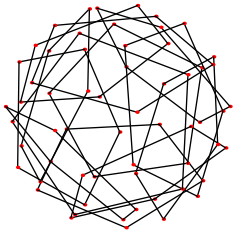
I C C 6, 2−2, 12×1×5



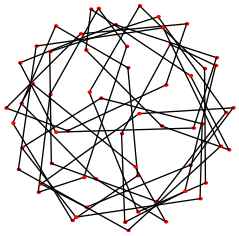
I C C 32, 2−2, 20×1×3



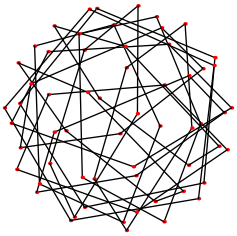
I C C 4, 2−2, 12×1×5



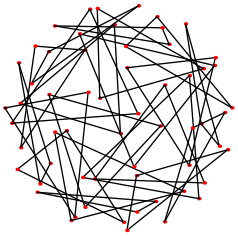
I C C 28, 2−2, 12×1×5



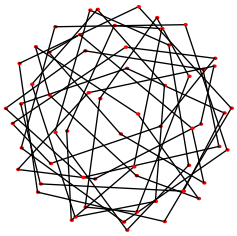
I C C 9, 2−2, 12×1×5



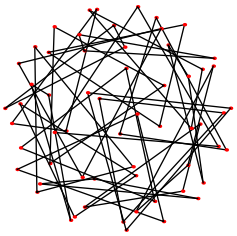
I C C 12, 2−2, 20×1×3



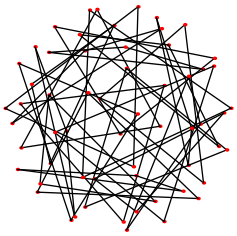
I C C 11, 2−2, 12×1×5



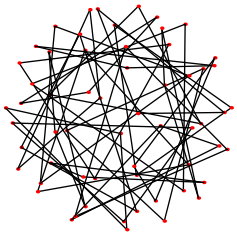
I C C 40, 2−2, 20×1×3



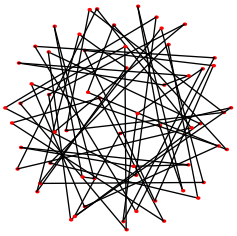
I C C 60, 2−2, 20×1×3



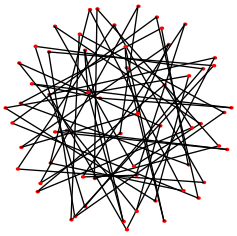
I C C 8, 2−2, 20×1×3



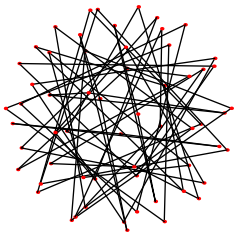
I C C 35, 2−2, 20×1×3



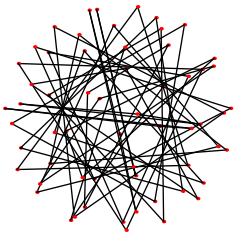
I C C 24, 2−2, 20×1×3



I C C 23, 2−2, 20×1×3



I C C 36, 2−2, 20×1×3



As you can see, some of these still don't qualify as interesting polypolyhedra. Yes, they are composed of multiple polyhedra; yes, they are at least two-valent at every vertex; but no, some of them do not consist of *linked* polyhedra.

But I couldn't dismiss an unlinked polypolyhedron out of hand. Take, for example, the polypolyhedron {T,C,C,2} (2nd row, first skeleton, a 4×1×3). Clearly, the polyhedra (triangles) are unlinked. But that's only for the particular value of $p^{(C)}$ that I chose as the initial orbital seed. I could have chosen any other value of $p^{(C)}$ that didn't fall on an axis of rotation; and it could be that one of those other values does indeed give a linked polypolyhedron (this actually turns out to be the case). Since I could vary the position of $p^{(C)}$ in two dimensions, there must be a continuous two-dimensional family of polypolyhedra associated just with the {T,C,C,2} polypolyhedron. In fact, there would be such a family for every {C,C} polypolyhedron.

Furthermore, consider the heteroorbital polypolyhedra. The vertices lie in two orbits, both of which correspond to sets of axes of rotation of the polypolyhedron. But I could choose the radial distances of those orbits independently, or if we neglect overall scale, I could vary the ratio of the orbital distances. Thus, there must be a continous 1-dimensional family of heteroorbital polypolyhedra.

So, I wasn't out of the woods yet. But the possiblities were down to a manageable number. The figures above represent all possible polypolyhedra. They fall naturally into three groups.

The first group is the homoorbital non-{C,C} polypolyhedra. There are only 4 of these, and apart from a constant scaling factor and mirror images, they are unique. They are:

```
1     -Polypolyhedron(O E1 E1 5, 2-2, 4×1×3)-
2     -Polypolyhedron(I E1 E1 5, 2-2, 6×1×5)-
3     -Polypolyhedron(I E1 E1 24, 2-2, 10×1×3)-
4     -Polypolyhedron(I F1 F1 23, 3-3, 5×4×3)-
```

The second group is the heteroorbital polypolyhedra. There are 9 of these. They are:

```
1     -Polypolyhedron(T E1 E2 3, 2-2, 3×1×4)-
2     -Polypolyhedron(O E1 E2 2, 2-2, 4×1×6)-
3     -Polypolyhedron(O E1 F1 4, 2-3, 4×3×4)-
4     -Polypolyhedron(I V1 E1 4, 5-2, 6×5×4)-
5     -Polypolyhedron(I E1 E2 6, 2-2, 6×1×10)-
6     -Polypolyhedron(I E1 E2 20, 2-2, 10×1×6)-
7     -Polypolyhedron(I E1 F1 6, 2-3, 5×4×6)-
8     -Polypolyhedron(I E1 F1 7, 2-3, 10×3×4)-
9     -Polypolyhedron(I F1 F2 7, 3-3, 5×6×4)-
```

But, as I said above, each of these is really a family, characterized by a 1-dimensional parameter.

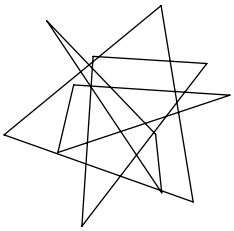
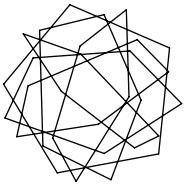
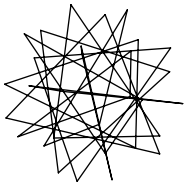
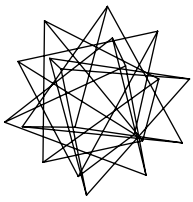
The third group is the homoorbital {C,C} polypolyhedra. There are 28 of these:

```
1       -Polypolyhedron(D C C 6, 2-2, 2×1×6)-
2       -Polypolyhedron(T C C 7, 2-2, 4×1×3)-
3       -Polypolyhedron(T C C 2, 2-2, 4×1×3)-
4       -Polypolyhedron(T C C 10, 2-2, 4×1×3)-
5       -Polypolyhedron(T C C 12, 2-2, 4×1×3)-
6       -Polypolyhedron(O C C 4, 2-2, 6×1×4)-
7       -Polypolyhedron(O C C 2, 2-2, 8×1×3)-
8       -Polypolyhedron(O C C 11, 2-2, 6×1×4)-
9       -Polypolyhedron(O C C 9, 2-2, 8×1×3)-
10      -Polypolyhedron(O C C 6, 2-2, 6×1×4)-
11      -Polypolyhedron(O C C 8, 2-2, 8×1×3)-
12      -Polypolyhedron(O C C 14, 2-2, 8×1×3)-
13      -Polypolyhedron(I C C 14, 2-2, 12×1×5)-
14      -Polypolyhedron(I C C 3, 2-2, 20×1×3)-
15      -Polypolyhedron(I C C 6, 2-2, 12×1×5)-
16      -Polypolyhedron(I C C 32, 2-2, 20×1×3)-
17      -Polypolyhedron(I C C 4, 2-2, 12×1×5)-
18      -Polypolyhedron(I C C 28, 2-2, 12×1×5)-
19      -Polypolyhedron(I C C 9, 2-2, 12×1×5)-
20      -Polypolyhedron(I C C 12, 2-2, 20×1×3)-
21      -Polypolyhedron(I C C 11, 2-2, 12×1×5)-
22      -Polypolyhedron(I C C 40, 2-2, 20×1×3)-
23      -Polypolyhedron(I C C 60, 2-2, 20×1×3)-
24      -Polypolyhedron(I C C 8, 2-2, 20×1×3)-
25      -Polypolyhedron(I C C 35, 2-2, 20×1×3)-
26      -Polypolyhedron(I C C 24, 2-2, 20×1×3)-
27      -Polypolyhedron(I C C 23, 2-2, 20×1×3)-
28      -Polypolyhedron(I C C 36, 2-2, 20×1×3)-
```

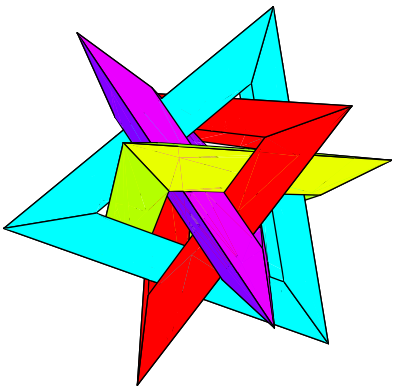
And each of these is characterized by a 2-dimensional set of parameters.

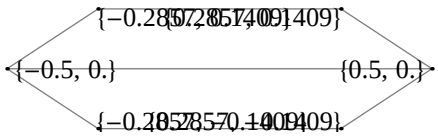
Well, let's continue. The first set (homoorbital non-{C,C}) are perhaps the easiest to handle because, other than an overall scale factor, there are no more degrees of freedom in each polypolyhedron. The four skeletons are plotted below.

O E1 E1 5, 2−2, 4×1×3 I E1 E1 5, 2−2, 6×1×5 I E1 E1 24, 2−2, 10×1×3 I F1 F1 23, 3−3, 5×4×3

The first three are: four interesecting triangles; six intersecting pentagons; ten intersecting triangles; and our old friend, five intersecting tetrahedra. These are all polypolyhedra that I identified in my first go-around at this problem. Using the same procedure I did before, I could solve for the pattern for an edge stick that can be assembled into the polypolyhedron. The edge sticks and the folded examples are shown below.

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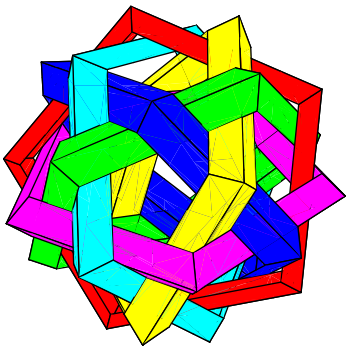

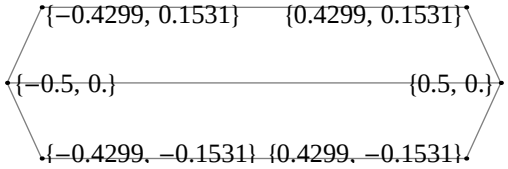
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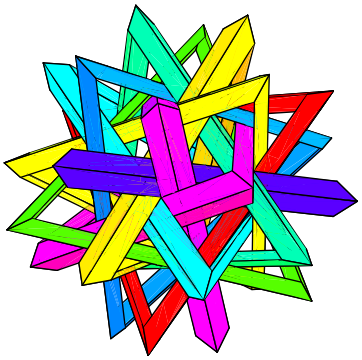
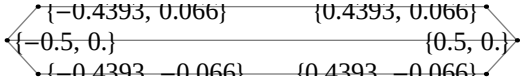
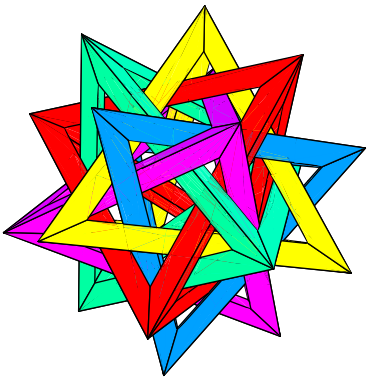
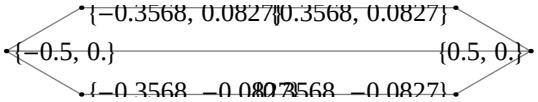
{−0.5, 0.}
{0.5, 0.}
{−0.4299, −0.1531}
{0.4299, −0.1531}




{−0.4393, 0.066}
{0.4393, 0.066}
{−0.5, 0.}
{0.5, 0.}





Life gets a bit more interesting when we turn to the heteroorbital polypolyhedra. The vertices come in two distinct orbits. Every vertex within a single orbit lies at the same orbital distance, but the orbital distances can vary completely independently.

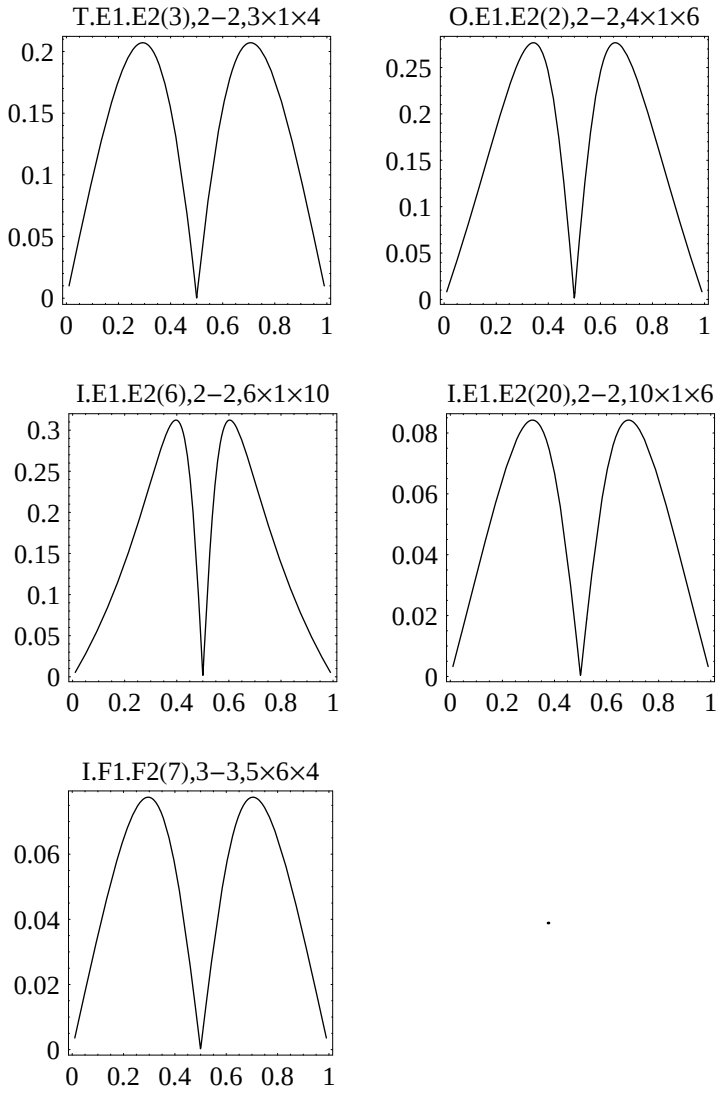
Any given vertex has its location defined by three coordinates: $\{x,y,z\}$, in Cartesian coordinates, or $\{\rho,\theta,\phi\}$ in polar coordinates. The angles θ and ϕ are fixed by the orbit — the point must lie on an axis of rotation — but the radial coordinate ρ can vary. Since we have two sets of vertices, we have two radial parameters ρ_1 and ρ_2, which can be freely chosen.

However, if we further stipulate that the polypolyhedron has an edge of unit length, then we no longer can choose the radial parameters independently; we can only choose their ratio ρ_1 / ρ_2. We can parameterize the two radial distances on a single parameter x that varies from 0 to 1. If we take $\rho_1 = f(x)\, x$ and $\rho_2 \equiv f(x)\,(1 - x)$, where the normalizing function *f(x)* is chosen to insure unit edge length for all values of x, then as we sweep x from 0 to 1, we will trace out all possible values of ρ_1 / ρ_2 from 0 to ∞.

Now, let's introduce a function $d_{\min}(x)$, which is defined as the minimum distance between the seed edge and any other edge in the polypolyhedron. If we plot this function, it will be nonzero for a valid polypolyhedron, which is the case for most values of x. However, at some points, it drops to zero. The zero values occur when the seed edge intersects one or more other edges of the polypolyhedron. As x moves through one of these zero values, the linkage between a polyhedron containing the seed edge and a polyhedron containing the intersecting edge changes. That means that the polypolyhedra obtained for x values on either size of a zero of $d_{\min}(x)$ are topologically distinct. Although they have the same number of polyhedra, faces, vertices, and vertex valencies, the polyhedra are linked in different ways.

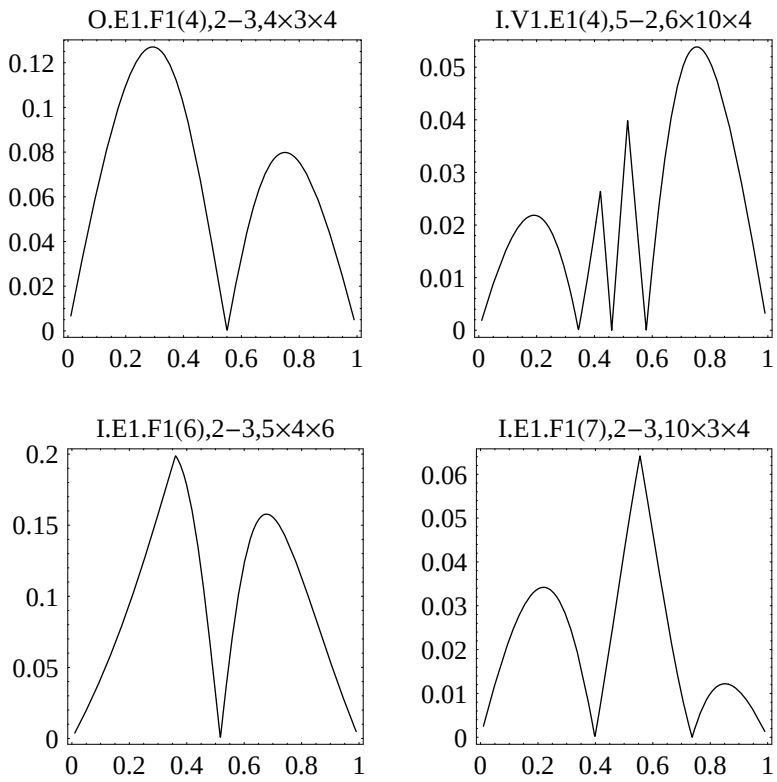
Now, we can divide the 9 heteroorbital polypolyhedra into two groups: those in which the two orbits are of the same basic type but different radial distances (e.g., {E1,E2} or {F1,F2}), and those in which the two orbits are entirely different (e.g., {E1, F1}). We'll call the former — where the two types of vertex lie in the same orbit but at different orbital distances — quasihomoorbital polypolyhedra. Quasihomoorbital polypolyhedra have a basic symmetry with respect to the variable x. The polypolyhedron obtained by making the substitution $x \to (1 - x)$ is the same polypolyhedron, but with the roles of E1 and E2 (or F1 and F2) reversed.

Thus, if we plot $d_{\min}(x)$ for the five quasihomoorbital polypolyhedra we indeed find that $d_{\min}(x)$ is symmetric about $x = 0.5$, where there is a zero.



The distance $d_{\min}(x)$ attains a maximum value between x=0 and 0.5. If we were to make the polypolyhedron from circular dowels, the value of x that gives the maximal value of $d_{\min}$ would give the largest diameter dowels. This also tends to give relatively wide angled sticks such as the edge unit I used for the 4 homoorbital non-{C,C} polypolyhedra. I solved for the optimum values of x for each of these 5 cases; this gave another 5 topologically distinct polypolyhedra. In the interest of space, I won't plot them all. But one of them — (I F1 F2 7) — is interesting. Though heteroorbital (2-uniform-vertex), all vertices are 3-valent — a property shared by Five Intersecting Tetrahedra. We'll come back to this one.

The other 4 heteroorbital types are truly heteroorbital: the vertices lie in two entirely different types of orbits. I plotted $d_{\min}(x)$ for each of these as well.



Now, there is no more symmetry with respect to the parameter x, so each lobe of each plot is a topologically distinct polypolyhedron. That means that there are 2 types of (O E1 F1,4), 4 of (I V1 E1 4), 2 of (I E1 F1 6), and 3 of (I E1 F1 7), for a total of 11 topologically distinct polypolyhedra. Add these to the 9 we found so far, and we're up to 20.

Now, things start to get hairy. I'd found 28 {C,C} orbital polypolyhedra — but those 28 were found for a single orbital seed point $p^{(C)}$. So far, so good. But that point can vary in two dimensions. Every possible value in the 2-D continuum of $p^{(C)}$ can be mapped to each of the 28 candidates.

But then I looked again at the listing, and there weren't that many after all. I could divide them up into 6 groups:

```
1     -Polypolyhedron(D C C 6, 2-2, 2×1×6)-

2     -Polypolyhedron(T C C 7, 2-2, 4×1×3)-
3     -Polypolyhedron(T C C 2, 2-2, 4×1×3)-
4     -Polypolyhedron(T C C 10, 2-2, 4×1×3)-
5     -Polypolyhedron(T C C 12, 2-2, 4×1×3)-

6      -Polypolyhedron(O C C 4, 2-2, 6×1×4)-
8      -Polypolyhedron(O C C 11, 2-2, 6×1×4)-
10     -Polypolyhedron(O C C 6, 2-2, 6×1×4)-

7      -Polypolyhedron(O C C 2, 2-2, 8×1×3)-
9      -Polypolyhedron(O C C 9, 2-2, 8×1×3)-
11     -Polypolyhedron(O C C 8, 2-2, 8×1×3)-
12     -Polypolyhedron(O C C 14, 2-2, 8×1×3)-
```

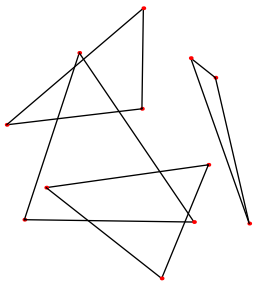
```
13      -Polypolyhedron(I C C 14, 2-2, 12×1×5)-
15      -Polypolyhedron(I C C 6, 2-2, 12×1×5)-
17      -Polypolyhedron(I C C 4, 2-2, 12×1×5)-
18      -Polypolyhedron(I C C 28, 2-2, 12×1×5)-
19      -Polypolyhedron(I C C 9, 2-2, 12×1×5)-
21      -Polypolyhedron(I C C 11, 2-2, 12×1×5)-

14      -Polypolyhedron(I C C 3, 2-2, 20×1×3)-
16      -Polypolyhedron(I C C 32, 2-2, 20×1×3)-
20      -Polypolyhedron(I C C 12, 2-2, 20×1×3)-
22      -Polypolyhedron(I C C 40, 2-2, 20×1×3)-
23      -Polypolyhedron(I C C 60, 2-2, 20×1×3)-
24      -Polypolyhedron(I C C 8, 2-2, 20×1×3)-
25      -Polypolyhedron(I C C 35, 2-2, 20×1×3)-
26      -Polypolyhedron(I C C 24, 2-2, 20×1×3)-
27      -Polypolyhedron(I C C 23, 2-2, 20×1×3)-
28      -Polypolyhedron(I C C 36, 2-2, 20×1×3)-
```

Setting aside the dihedral group for the moment, consider the four tetrahedral types. Each is composed of 4 triangles (4×1×3) and the edges have tetrahedral rotation symmetry. Which means that they are very similar. In fact, each of the 4 can be obtained from any of the others by rotating the four triangles about their centers and/or translating the triangles with respect to the origin.

But this is a general transformation: any possible arrangement of four triangles with tetrahedral rotation symmetry should be obtainable by the same transformation. So, we can drop all but one of the (4×1×3) {C,C} configurations and look at how it varies under this rotation/translation transformation.

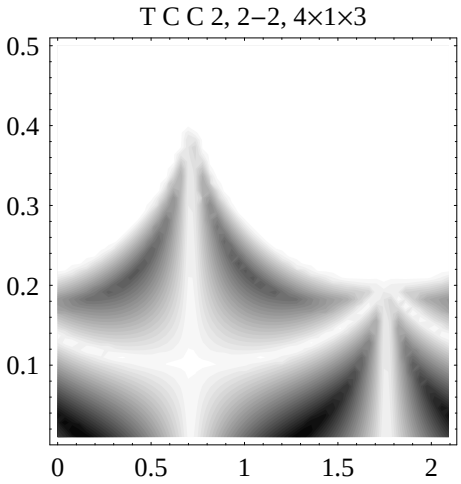
Specifically, let's take the (T C C 2) polypolyhedron, whose skeleton looks like this:

T C C 2, 2−2, 4×1×3



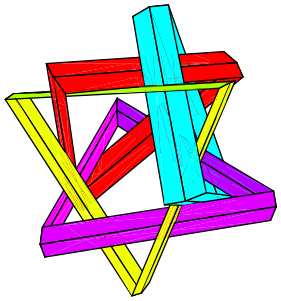
Now, imagine rotating each triangle by an angle ϕ and translate it radially so that its centroid is a distance ρ from the origin. For every $\{\phi,\rho\}$ pair, we get a unique arrangement of triangles — and for some $\{\phi,\rho\}$, we get every possible arrangement of equilateral triangles with tetrahedral symmetry.

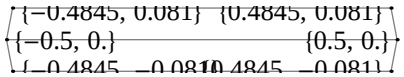
Now, let's call up our old friend d_{min}, the shortest distance between the reference edge and any other edge. I plot it now in two dimensions versus ϕ and ρ, plotting only those values for which the triangles are linked (they clearly are not for the example shown above). That plot is as follows:



The horizontal axis is rotation angle (ϕ) and the vertical axis is distance from the center (ρ). Darker regions correspond to greater distances between the seed edge and any other edge. Obviously, the function is periodic in ϕ with period 2π (imagine the plot wrapped around like a vertical cylinder). As with the heteroorbital polypolyhedra, crossing a zero of $d_{\text{min}}(\phi, \rho)$ means the topological linkage has changed; thus, each shaded region corresponds to a topologically distinct polypolyhedron. It is apparent from the plot that there are 4 such regions (the right and left edge of the plot should be thought of as joined). Two of them are mirror images of the other two. (Mirror image polypolyhedra lie symmetrically about the obvious vertical symmetry lines of the plot.) So, not counting mirror images, there are two distinct tetrahedral {C,C} polypolyhedra.

The one at $\{\phi\approx1.3,\rho\approx0\}$ turns out to be identical to the (4×1×3) we already found, the homoorbital (O E1 E2 2). But there is another tetrahedral polypolyhedron, located at $\{\phi\approx1.03,\rho\approx.203\}$, which is a different arrangement of four triangles. It is shown here (along with the unfolded version of the origami edge strut).

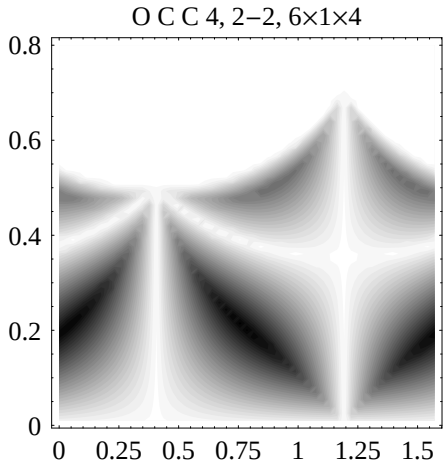




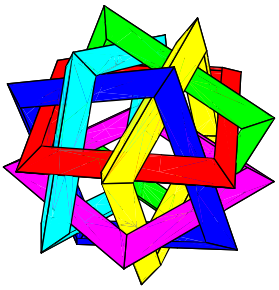
And that completes the tetrahedral enumeration (whew!).

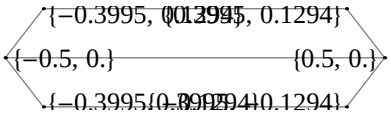
Turning back to the dihedral case, if we do the same plot for (D C C 6), we will find that there are no values of ϕ and ρ that give linked polygons: there are no polypolyhedra with dihedral symmetry.

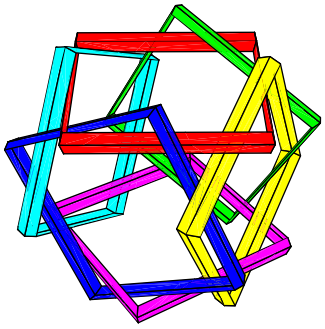
Next come the two possibilities for octahedral {C C} orbits. We can let (O C C 4) stand in for all possible 6×1×4 polypolyhedra. The plot of $d_{\text{min}}(\phi, \rho)$ is this:

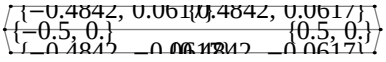


This reveals that there are two distinct octahedral 6×1×4 polypolyhedra, which I show below.



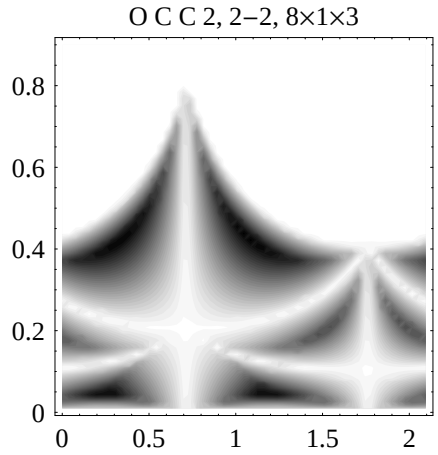






I should point out that there are many possible types of origami edge unit for each polypolyhedron. I've shown an edge unit analogous to the Ow unit used in FIT.

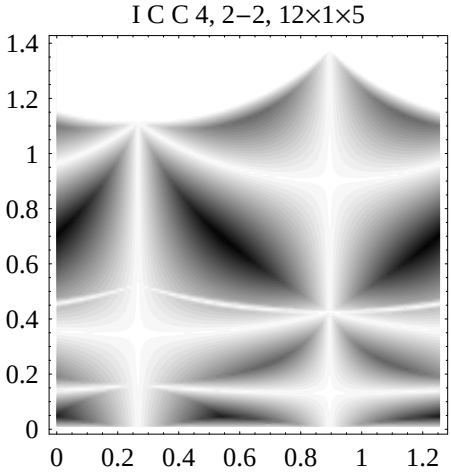
The other octahedral {C,C} type is the 8×1×3, whose plot of $d_{\min}$ is:



Now, if you look closely, you'll see that there are 3 topologically distinct polypolyhedra (plus their mirror images). In the interest of space, I will not plot them here.

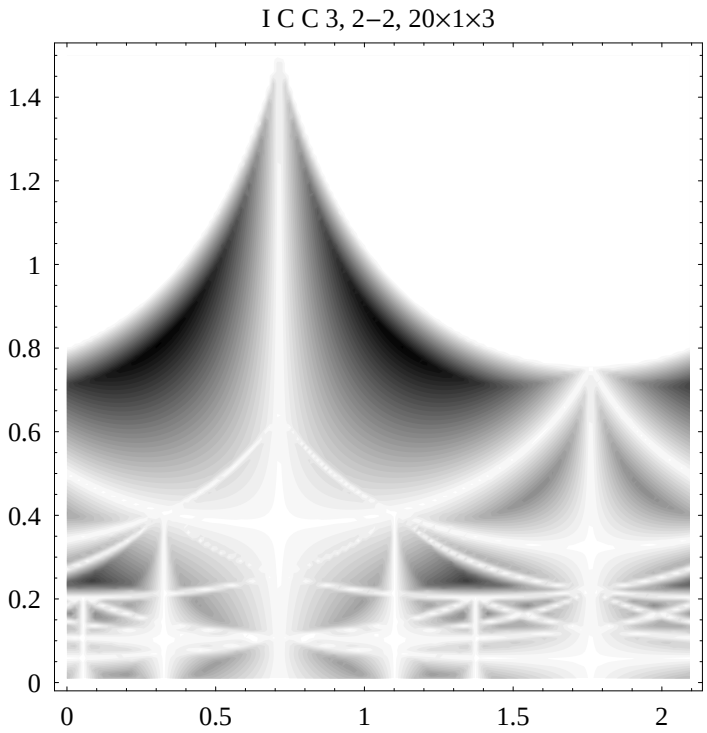
It is interesting to note that the five families of {C,C} polypolyhedra each correspond to one of the Platonic solids.

Next, we move to the icosahedral group. The 12×1×5 family yields the following plot:



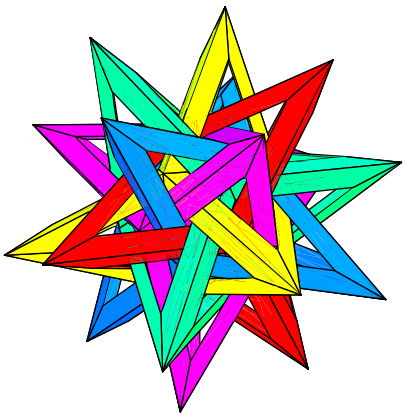
There are five topologically distinct arrangements (don't overlook the narrow sliver around $\rho \approx .45$), plus their mirror images. Again, I won't plot them all in the interest of space.

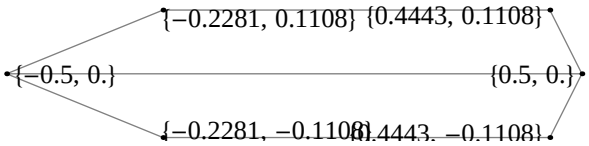
Last, but certainly not least, comes the 20×1×3 set. You might have noticed the number of polypolyhedra per family creeping upward as we went: 2, 2, 3, 5…, which leads one to suspect that the 20×1×3 would be the largest group yet: perhaps 8? 10? 15? In fact, the icosahedral 20×1×3 group leads to an astonishing total of 23 different varieties (plus mirror images, of course), which you can see from a careful examination of the $d_{\text{min}}(\phi, \rho)$ plot:



When I put together all of the polypolyhedra of the various types, I find a total of exactly 54 topologically distinct polypolyhedra, not counting rotations and reflections. Nearly half of them are 20×1×3. All of them are excellent candidates for origami implementation.

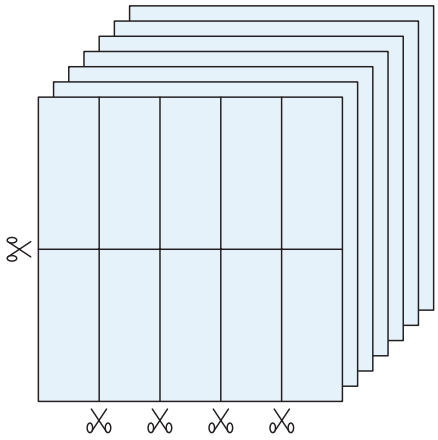
But one of them is more excellent than others. Out of all polypolyhedra, all but two have at least some 2-valent vertices. The two exceptions are the 5×4×3, a.k.a. Five Intersecting Tetrahedra, and one other that we encountered back in the quasihomoorbital polypolyhedra — the (I F1 F2 7) polypolyhedron, whose structure is 5×6×4, and whose implementation with a dihedral origami edge unit is the following:



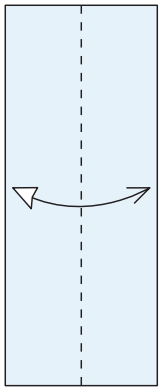


This figure can be constructed from 60 units. Each unit joins three others at either end. In working out the folding sequence for the unit, several fortuitous coincidences of proportion occur, making a fairly elegant and simple folding sequence for the unit.

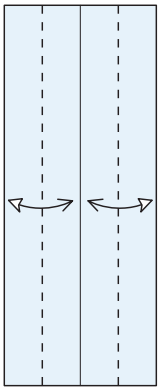
Each of the polyhedra is a tetrahedrally distorted hexahedron — in other words, a cube, with four tetrahedrally arranged corners stretched outward. 12 units are assembled into each of the 5 hexahedra. All but the first must be built linked with the others.



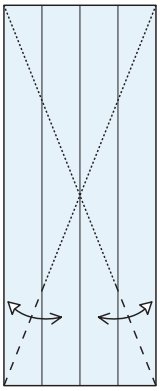
1. This modular uses sixty rectangles, each in the proportions of 1:2.5. You can obtain these from six squares by cutting each square into half one way and into fifths the other.



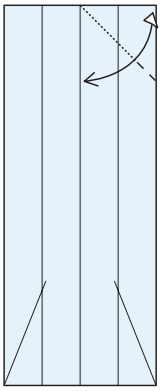
2. Take one of the rectangles. Fold it in half the long way and unfold.



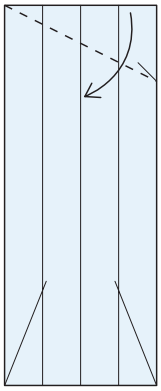
3. Fold the sides in to the center and unfold.



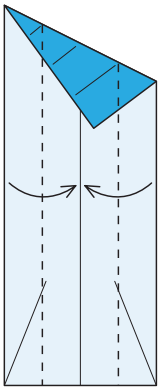
4. Fold the paper along the diagonal corner to corner, but only make the crease sharp along the bottom fourth.



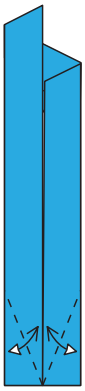
5. Fold the top right corner down to the center line, but only make the crease sharp along the right edge.



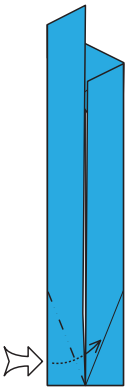
6. Fold the top right corner down. The crease hits the right edge at the same place as the crease from step 5.



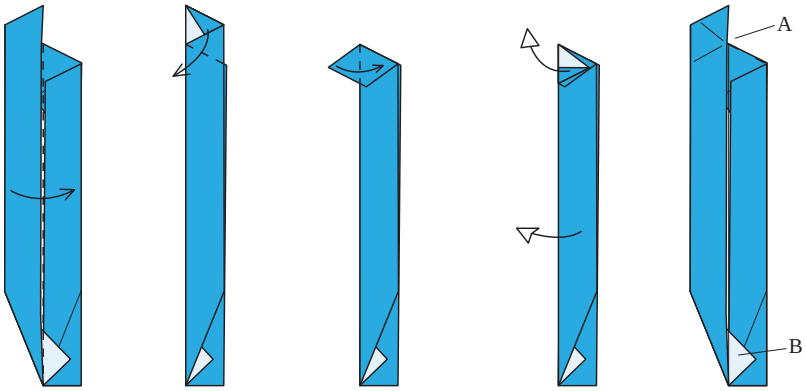
7. Fold the sides in along the existing creases.



8. Fold and unfold through both layers on the existing creases.



9. Reverse-fold the bottom edge.



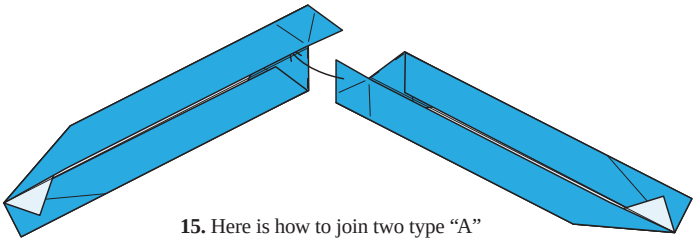
10. Fold the unit in half.

11. Fold the top flap down so that the crease lines up with the edge behind.

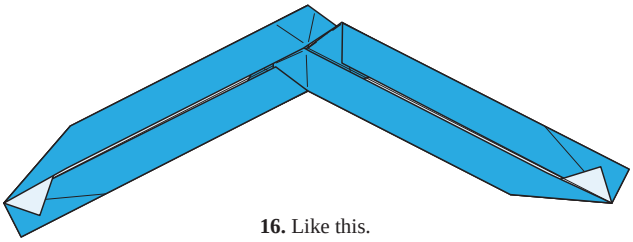
12. Fold the tip back to the right along a vertical crease.

13. Unfold to step 10.

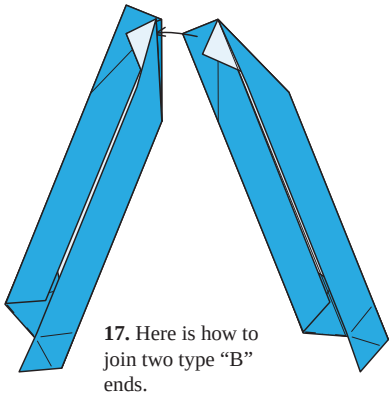
14. Finished unit. Make 59 more units. Each unit has an “A” end and a “B” end.



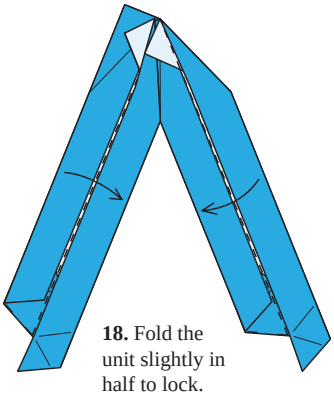
15. Here is how to join two type “A” ends.



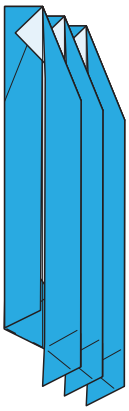
16. Like this.



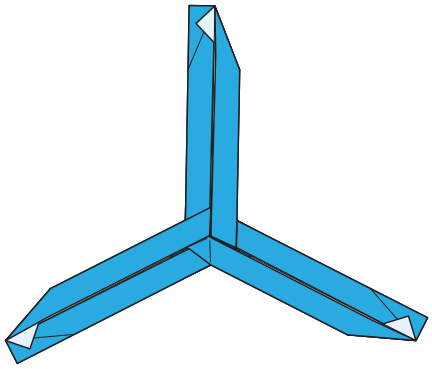
17. Here is how to join two type “B” ends.



18. Fold the unit slightly in half to lock.

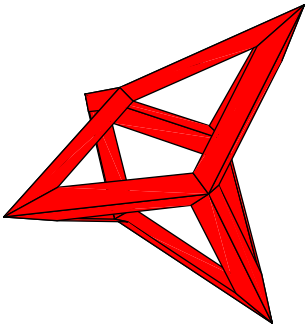


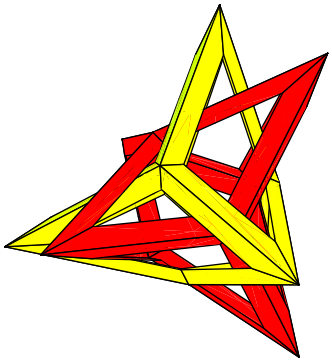
19. Take three units and join them at their "B" ends.

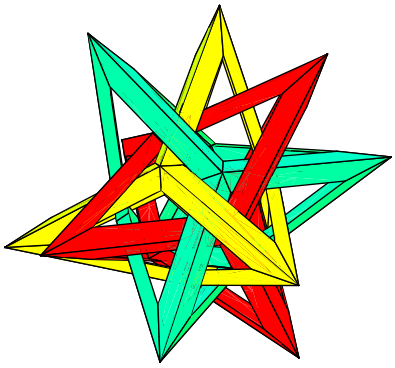


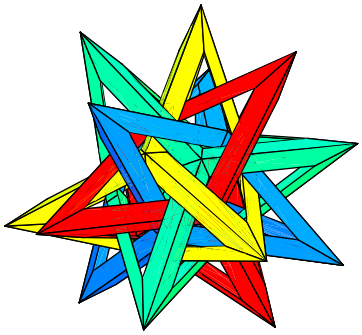
20. Take four groups of three units and join them at their "A" ends to form a tetrahedrally distorted skew rhombic hexahedron.

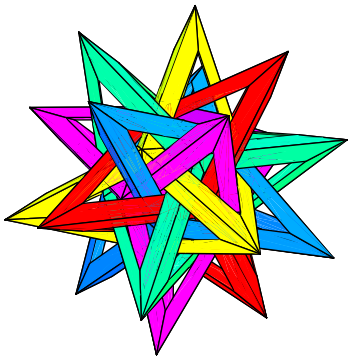
The assembly of the units poses some interesting challenges. Here is the sequential assembly of the five hexahedral units.











This seems like a nice place to wrap up this subject.

An epilog: I showed some of this work to a few people at the Gathering 4 Gardner 4 meeting in Atlanta in February. One of them, Stan Isaacs, asked me, "Oh, are you making all the ones in the book?" Ummmm…what book might that be? After a bit of prodding, he came up with a name and title: *Orderly Tangles*, by Alan Holden. I immediately went and bought a copy (used bookstores on the Web are God's gift to mankind) to find out what he was talking about. Turns out that some of what I've been doing has a name. Holden — a Bell Labs scientist at the time he wrote the book (1980) — coined the name "polylink" for three-dimensional linked arrangements of regular polygons with uniform vertices — in my words, the homoorbital $N \times 1 \times M$ polypolyhedra. Holden built his polylinks from wooden dowels, 1980 being the dark ages before *Mathematica*, and I was interested to see photographs of many of "my" polypolyhedra in his book.

But not all, as it turns out. To my knowledge (though I haven't yet done a truly thorough search), no one has enumerated all of the polypolyhedron; this work is the first *complete* enumeration of polypolyhedra and regular polylinks (for example, Holden only identifies 12 in the 20×1×3 family). So what started out as a fairly simple question turned out to lead to some pretty amazing — and perhaps even new — mathematics. And origami, too.